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A NOTE ON A CERTAIN RETARDED INTEGRAL INEQUALITY

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Abstract. In the present note we establish explicit bound on a certain retarded integral inequality which can be used as a tool in the study of retarded differential and integral equations.

1. Introduction

The integeral inequalities which provide explicit bounds on the unknown functions play a fundamental role in the development of the theory of differential and integral equations (see [2, 3, 4]). It is hardly imagine these theories without such inequalities. In 1972 Ahmedov, Jakubov and Veisov [1] obtained the upper bound on the following inequality (see [3, Theorem 33]):

$$u(t) \leq f(t) + \int_{t_0}^{\phi_1(t)} a_1(s) F_1(u(s)) ds + \int_{t_0}^{\phi_2(t)} a_2(s) F_2(u(s)) ds, \tag{1.1}$$

under some suitable conditions on the functions involved in (1.1). The bound obtained on (1.1) can be used to study the qualitative behavior of solutions of certain retarded differential and integral equations. In fact the bound established on (1.1) is based on the solution of a certain initial value problem (see [3, p.18]) which basically involves the comparison principle and usually difficult to calculate explicitly. The main purpose of this note is to establish explicit bound on the general version of (1.1) which can be used more effectively to study the qualitative behavior of solutions of certain classes of retarded differential and ingegral equations. The two independent variable generalization of the main result and an application is also given.

2. Statement of Results

In what follows, R denotes the set of real numbers, $R_+ = [0, \infty)$, $I = [t_0, \alpha)$, $J_1 = [x_0, \alpha)$, $J_2 = [y_0, \beta)$ are the given subsets of R, $\Delta = J_1 \times J_2$, $E = \{(t, s) \in I^2 : t_0 \leq s \leq t < \alpha\}$ and $'$ denotes the derivative. The first order partial derivatives of a function $z(x, y)$, x, $y \in R$ with respect to x and y are denoted by $D_1 z(x, y)$ and $D_2 z(x, y)$ respectively.

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For some suitable functions defined on the respective domains of their definitions,first we give the following notations used to simplify the details of presentation:

$$\begin{aligned}
&H[t,m;\phi_1,a_1,p_1;\phi_2,a_2,p_2]\\
&=\int_{\phi_1(t_0)}^{\phi_1(t)} a_1(s)p_1(m(s))ds+\int_{\phi_2(t_0)}^{\phi_2(t)} a_2(s)p_2(m(s))ds,\\
&F[x,y,m;\phi_1,\psi_1,a_1,p_1;\phi_2,\psi_2,a_2,p_2]\\
&=\int_{\phi_1(x_0)}^{\phi_1(x)}\int_{\psi_1(y_0)}^{\psi_1(y)} a_1(s,t)p_1(m(s,t))dtds+\int_{\phi_2(x_0)}^{\phi_2(x)}\int_{\psi_2(y_0)}^{\psi_2(y)} a_2(s,t)p_2(m(s,t))dtds.
\end{aligned}$$

Our main result is given in the following theorem.

Theorem 1. *Let u, f, b, a_1, $a_2 \in C(I,R_+)$, ϕ_1, $\phi_2 \in C^1(I,I)$ be nondecreasing with $\phi_1(t)\le t$, $\phi_2(t)\le t$ on I. For $i=1,2$, let $g_i \in C(R_+,R_+)$ be nondecreasing, subadditive and submultiplicative functions with $g_i(u)>0$ for $u>0$ and for $t\in I$,*

$$u(t)\le f(t)+b(t)H[t,u;\phi_1,a_1,g_1;\phi_2,a_2,g_2], \tag{2.1}$$

then for $t_0\le t\le t_1$, t, $t_1\in I$

(i) *in case $g_2(u)\le g_1(u)$,*

$$u(t)\le f(t)+b(t)G_1^{-1}[G_1(A(t))+H[t,b;\phi_1,a_1,g_1;\phi_2,a_2,g_2]], \tag{2.2}$$

(ii) *in case $g_1(u)\le g_2(u)$,*

$$u(t)\le f(t)+b(t)G_2^{-1}[G_2(A(t))+H[t,b;\phi_1,a_1,g_1;\phi_2,a_2,g_2]], \tag{2.3}$$

where

$$A(t)=H[t,f;\phi_1,a_1,g_1;\phi_2,a_2,g_2], \tag{2.4}$$

and for $i=1$, 2, G_i^{-1} are the inverse functions of

$$G_i(r)=\int_{r_0}^{r}\frac{ds}{g_i(s)},\quad r>0,\ r_0>0, \tag{2.5}$$

and $t_1\in I$ is chosen so that

$$G_i(A(t))+H[t,b,\phi_1,a_1,g_1;\phi_2,a_2,g_2]\in Dom(G_i^{-1}),$$

respectively, for all t lying in the interval $t_0\le t\le t_1$.

Remark 1. We note that in [1] the authors have given the upper bound on (1.1) which depends on the continuous solution of a certain initial value problem for first order differential equation. The bound obtained on the general version of (1.1) in Theorem 1 is explicit and it is more convenient in applications. However, the conditions required here on the functions involved in (2.1) are different from those of given in [1].

In the following theorem we establish two independent variable version of Theorem 1 which can be used in certain applications.

Theorem 2. *Let u, f, b, a_1, $a_2 \in C(\Delta, R_+)$ and ϕ_1, $\phi_2 \in C^1(J_1, J_1)$, ψ_1, $\psi_2 \in C^1(J_2, J_2)$ be nondecreasing with $\phi_i(x) \leq x$ on J_1, $\psi_i(y) \leq y$ on J_2 for $i = 1, 2$. Let $g_i(u)$, $i = 1, 2$ be as in Theorem 1 and for $(x, y) \in \Delta$,*

$$u(x,y) \leq f(x,y) + b(x,y)F[x,y,u;\phi_1,\psi_1,a_1,g_1;\phi_2,\psi_2,a_2,g_2], \tag{2.6}$$

then for $x_0 \leq x \leq x_1$, $y_0 \leq y \leq y_1$; x, $x_1 \in J_1$, y, $y_1 \in J_2$

(i) *in case $g_2(u) \leq g_1(u)$,*

$$u(x,y) \leq f(x,y) + b(x,y)G_1^{-1}[G_1(B(x,y)) + F[x,y,b;\phi_1,\psi_1,a_1,g_1;\phi_2,\psi_2,a_2,g_2]], \tag{2.7}$$

(ii) *in case $g_1(u) \leq g_2(u)$,*

$$u(x,y) \leq f(x,y) + b(x,y)G_2^{-1}[G_2(B(x,y)) + F[x,y,b;\phi_1,\psi_1,a_1,g_1;\phi_2,\psi_2,a_2,g_2]], \tag{2.8}$$

where G_i, G_i^{-1} are as in Theorem 1 and

$$B(x,y) = F[x,y,f;\phi_1,\psi_1,a_1,g_1;\phi_2,\psi_2,a_2,g_2], \tag{2.9}$$

and $x_1 \in J_1$, $y_1 \in J_2$ are chosen so that for $i = 1, 2$

$$G_i(B(x,y)) + F[x,y,b;\phi_1,\psi_1,a_1,g_1;\phi_2,\psi_2,a_2,g_2] \in Dom(G_i^{-1}),$$

for all x, y lying in $x_0 \leq x \leq x_1$ and $y_0 \leq y \leq y_1$ respectively.

Remark 2. We note that the inequalities established in Theorem 2 parts (b_2), (b_3) and Theorem 4 parts (d_2), (d_3) in [5] can be extended very easily in the framework of the inequalities in Theorems 1 and 2 given here. We also note that the inequality established in Theorem 2 given above can be extended to functions involving many independent variables.Since these translations are quite straightforward in view of the results given above, we leave it for the readers to fill in where needed. For explicit bounds on inequalities involving functions of more than two independent variables, but without retarded arguments, see [2, 4].

3. Proofs of Theorems 1 and 2

From the hypotheses in Theorem 1 we observe that $\phi_1'(t) \geq 0$, $\phi_2'(t) \geq 0$ for $t \in I$. Define a function $z(t)$ by

$$\begin{aligned} z(t) &= H[t,u;\phi_1,a_1,g_1;\phi_2,a_2,g_2] \\ &= \int_{\phi_1(t_0)}^{\phi_1(t)} a_1(s)g_1(u(s))ds + \int_{\phi_2(t_0)}^{\phi_2(t)} a_2(s)g_2(u(s))ds. \end{aligned} \tag{3.1}$$

Then $z(t_0) = 0$ and (2.1) can be restated as

$$u(t) \le f(t) + b(t)z(t). \tag{3.2}$$

Using (3.2) in (3.1) and the hypotheses on g_1 and g_2 we have

$$\begin{aligned} z(t) &\le \int_{\phi_1(t_0)}^{\phi_1(t)} a_1(s)g_1(f(s) + b(s)z(s))ds + \int_{\phi_2(t_0)}^{\phi_2(t)} a_2(s)g_2(f(s) + b(s)z(s))ds \\ &\le A(t) + \int_{\phi_1(t_0)}^{\phi_1(t)} a_1(s)g_1(b(s))g_1(z(s))ds + \int_{\phi_2(t_0)}^{\phi_2(t)} a_2(s)g_2(b(s))g_2(z(s))ds. \end{aligned} \tag{3.3}$$

Let $T \in I$ be any arbitrary number. From(3.3), for $t_0 \le t \le T$ we have

$$z(t) \le A(T) + \int_{\phi_1(t_0)}^{\phi_1(t)} a_1(s)g_1(b(s))g_1(z(s))ds + \int_{\phi_2(t_0)}^{\phi_2(t)} a_2(s)g_2(b(s))g_2(z(s))ds. \tag{3.4}$$

Now assume that $A(T) > 0$ and let $g_2(u) \le g_1(u)$. Define a function $v(t)$ by the right hand side of (3.4). Then $v(t_0) = A(T)$, $z(t) \le v(t)$, $v(t)$ is nondecreasing for $t_0 \le t \le T$ and

$$\begin{aligned} v'(t) &= a_1(\phi_1(t))g_1(b(\phi_1(t)))g_1(z(\phi_1(t)))\phi_1'(t) + a_2(\phi_2(t))g_2(b(\phi_2(t)))g_2(z(\phi_2(t)))\phi_2'(t) \\ &\le a_1(\phi_1(t))g_1(b(\phi_1(t)))g_1(v(\phi_1(t)))\phi_1'(t) + a_2(\phi_2(t))g_2(b(\phi_2(t)))g_2(v(\phi_2(t)))\phi_2'(t) \\ &\le a_1(\phi_1(t))g_1(b(\phi_1(t)))g_1(v(t))\phi_1'(t) + a_2(\phi_2(t))g_2(b(\phi_2(t)))g_2(v(t))\phi_2'(t) \\ &\le [a_1(\phi_1(t))g_1(b(\phi_1(t)))\phi_1'(t) + a_2(\phi_2(t))g_2(b(\phi_2(t)))\phi_2'(t)]g_1(v(t)). \end{aligned} \tag{3.5}$$

From (2.5) and (3.5) we have

$$\begin{aligned} \frac{d}{dt}G_1(v(t)) &= \frac{v'(t)}{g_1(v(t))} \\ &\le [a_1(\phi_1(t))g_1(b(\phi_1(t)))\phi_1'(t) + a_2(\phi_2(t))g_2(b(\phi_2(t)))\phi_2'(t)]. \end{aligned} \tag{3.6}$$

By taking $t = s$ in (3.6) and integrating it with respect to s from t_0 to t for $t_0 \le t \le T$ we get

$$\begin{aligned} G_1(v(t)) \le G_1(A(T)) + \int_{t_0}^{t} &[a_1(\phi_1(s))g_1(b(\phi_1(s)))\phi_1'(s) \\ &+ a_2(\phi_2(s))g_2(b(\phi_2(s)))\phi_2'(s)]ds, \end{aligned} \tag{3.7}$$

for $t_0 \le t \le T$. Since $z(t) \le v(t)$ for $t_0 \le t \le T$ and $T \in I$ is arbitrary, from (3.7) we have

$$\begin{aligned} z(t) \le G_1^{-1}\Bigg[G_1(A(t)) + \int_{t_0}^{t} &[a_1(\phi_1(s))g_1(b(\phi_1(s)))\phi_1'(s) \\ &+ a_2(\phi_2(s))g_2(b(\phi_2(s)))\phi_2'(s)]ds\Bigg], \end{aligned} \tag{3.8}$$

for $t_0 \leq t \leq t_1$. By making the change of variable in the integral on the right side in (3.8) we have

$$z(t) \leq G_1^{-1}[G_1(A(t)) + H[t, b; \phi_1, a_1, g_1; \phi_2, a_2, g_2]], \tag{3.9}$$

for $t_0 \leq t \leq t_1$. The conclusion (2.2) is now clear from (3.2) and (3.9). The subinterval $t_0 \leq t \leq t_1$ is obvious.

If $A(T)$ in (3.4) is nonnegative, we carry out the above procedure with $A(T) + \varepsilon$ instead of $A(T)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit $\varepsilon \to 0$ to obtain (2.2). The proof of the case when $g_1(u) \leq g_2(u)$ can be completed similarly. This completes the proof of Theorem 1.

From the hypotheses of Theorem 2 we observe that for $i = 1, 2$, $\phi_i'(x) \geq 0$, $x \in J_1$; $\psi_i'(y) \geq 0$, $y \in J_2$. Define a function $z(x, y)$ by

$$z(x, y) = F[x, y, u; \phi_1, \psi_1, a_1, g_1; \phi_2, \psi_2, a_2, g_2]. \tag{3.10}$$

Then $z(x_0, y) = z(x, y_0) = 0$ and (2.6) can be restated as

$$u(x, y) \leq f(x, y) + b(x, y)z(x, y). \tag{3.11}$$

Using (3.11) in (3.10) and making use of the hypotheses on g_1, g_2 we get

$$\begin{aligned} z(x, y) \leq B(x, y) &+ \int_{\phi_1(x_0)}^{\phi_1(x)} \int_{\psi_1(y_0)}^{\psi_1(y)} a_1(s, t) g_1(b(s, t)) g_1(z(s, t)) dt ds \\ &+ \int_{\phi_2(x_0)}^{\phi_2(x)} \int_{\psi_2(y_0)}^{\psi_2(y)} a_2(s, t) g_2(b(s, t)) g_2(z(s, t)) dt ds. \end{aligned} \tag{3.12}$$

The rest of the proof can be completed by closely looking at the proof of Theorem 1 given above and following the proofs of similar results given in [5] and [4] with suitable changes. Further details are omitted here.

4. An Application

In this section we present an application of Theorem 1 to obtain the estimate on the solution of retarded Volterra integral equation of the form

$$x(t) = p(t) + \int_{t_0}^{t} F_1(t, s, x(s - h_1(s))) ds + \int_{t_0}^{t} F_2(t, s, x(s - h_2(s))) ds, \tag{4.1}$$

for $t \in I$, where $p \in C(I, R)$ and for $i = 1, 2$, $F_i \in C(E \times R, R)$, $h_i \in C^1(I, I)$ be nonincreasing functions with $t - h_i(t) \geq 0$, $h_i'(t) < 1$. Here we note that the existence proofs for the solutions of (4.1) show either that the operator S defined by the right hand side of (4.1) is a contration (in which case one also has uniqueness) or S is compact and continuous on a suitable subspace of the space of continuous functions.

The following theorem deals with the estimate on the solution of equation (4.1).

Theorem 3. *Suppose that the functions p, F_i in* (4.1) *satisfy the conditions*

$$|p(t)| \le f(t), \tag{4.2}$$

$$|F_i(t, s, u_i)| \le b(t) a_i(s) g_i(|u_i|), \tag{4.3}$$

for $i = 1$, 2, where f, b, a_i, g_i, are as in Theorem 1. *For $i = 1$, 2 let*

$$M_i = \max_{t \in I} \frac{1}{1 - h_i'(t)}, \tag{4.4}$$

and G_i, G_i^{-1} be as in Theorem 1. *If $x(t)$ is any solution of equation* (4.1) *on I, then for $t_0 \le t \le t_1$, t, $t_1 \in I$,*

(i) *in case $g_2(u) \le g_1(u)$,*

$$|x(t)| \le f(t) + b(t) G_1^{-1}[G_1(\bar{A}(t)) + H[t, b; \phi_1, \bar{a}_1, g_1; \phi_2, \bar{a}_2, g_2]], \tag{4.5}$$

(ii) *in case $g_1(u) \le g_2(u)$,*

$$|x(t)| \le f(t) + b(t) G_2^{-1}[G_2(\bar{A}(t)) + H[t, b; \phi_1, \bar{a}_1, g_1; \phi_2, \bar{a}_2, g_2]], \tag{4.6}$$

where for $i = 1$ 2; $\phi_i(t) = t - h_i(t)$, $\bar{a}_i(\sigma) = M_i a_i(\sigma + h_i(s))$, for σ, $s \in I$,

$$\bar{A}(t) = H[t, f; \phi_1, \bar{a}_1, g_1; \phi_2, \bar{a}_2, g_2], \tag{4.7}$$

and $t_1 \in I$ is chosen so that

$$G_i(\bar{A}(t)) + H[t, b; \phi_1, \bar{a}_1, g_1; \phi_2, \bar{a}_2, g_2] \in Dom(G_i^{-1}),$$

respectively, for all t lying in $t_0 \le t \le t_1$.

Proof. Let $x(t)$ be a solution of (4.1). Using the fact that $x(t)$ is a solution of (4.1) and the conditions (4.2), (4.3) we get

$$|x(t)| \le f(t) + b(t) \left[\int_{t_0}^{t} a_1(s) g_1(|x(s - h_1(s))|) ds + \int_{t_0}^{t} a_2(s) g_2(|x(s - h_2(s))|) ds \right]. \tag{4.8}$$

By making the change of variables in the integrals on the right hand side of (4.8) and using (4.4) we have

$$|x(t)| \le f(t) + b(t) \left[\int_{\phi_1(t_0)}^{\phi_1(t)} \bar{a}_1(\sigma) g_1(|x(\sigma)|) d\sigma + \int_{\phi_2(t_0)}^{\phi_2(t)} \bar{a}_2(\sigma) g_2(|x(\sigma)|) d\sigma \right]. \tag{4.9}$$

Now a suitable application of Theorem 1 yields the desired bounds in (4.5) and (4.6). The proof is complete.

Remark 3. It is easy to observe that the inequality established in Theorem 2 can be used to obtain the bound on the solution of the retarded Volterra ingegral equation of the form

$$\begin{aligned} z(x,y) = r(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} F_1(x,y,s,t,z(s-h_1(s),t-g_1(t)))dtds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} F_2(x,y,s,t,z(s-h_2(s),t-g_2(t)))dtds, \end{aligned} \tag{4.10}$$

under some suitable conditions on the functions involved in (4.10). For similar applications, see [5].

References

[1] K. T. Ahmedov, M. A. Jakubov and J. A. Veisov, *Some integral inequalities*, Izv. Akad. Nauk UzSSR **1**(1972), 18-24 (in Russian).

[2] D. Bainov and P. Simeonov, Integral Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, 1992.

[3] S. S. Dragomir, Some Gronwall Type Inequalities and Applications, Nova Science Publishers, Inc. New York, 2003.

[4] B. G. Pachpatte, Inequalities for Differential and Integral Equations, Academic Press, New York, 1998.

[5] B. G. Pachpatte, *Explicit bounds on certain integral inequalities*, J. Math. Anal. Appl. **267**(2002), 48-61.

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