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COEFFICIENT INEQUALITIES FOR STARLIKENESS AND CONVEXITY

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Abstract. For an analytic function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ satisfying the inequality $\sum_{n=2}^{\infty}n(n-1)|a_n|\leq\beta$, sharp bound on β is determined so that f is either starlike or convex of order α. Several other coefficient inequalities related to certain subclasses are also investigated.

1. Introduction

Let $\mathscr{A}$ be the class of analytic functions in $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, normalized by $f(0)=0$ and $f'(0)=1$. A function $f\in\mathscr{A}$ has a Taylor's series expansion of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

For $0\leq\alpha<1$, let $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$ be the subclasses of $\mathscr{A}$ consisting respectively of starlike functions of order α and convex functions of order α. These functions are known to be univalent, and are defined analytically by

$$\mathscr{S}^*(\alpha):=\left\{f\in\mathscr{A}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\right\},$$

and

$$\mathscr{C}(\alpha):=\left\{f\in\mathscr{A}:\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\right\}.$$

The classes $\mathscr{S}^*:=\mathscr{S}^*(0)$ and $\mathscr{C}:=\mathscr{C}(0)$ are the familiar classes of starlike and convex functions. Closely related are the classes of functions

$$\mathscr{S}^*_\alpha:=\left\{f\in\mathscr{A}:\left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha\right\},$$

and

$$\mathscr{C}_\alpha:=\left\{f\in\mathscr{A}:\left|\frac{zf''(z)}{f'(z)}\right|<1-\alpha\right\}.$$

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Note that $\mathscr{S}^*_\alpha \subseteq \mathscr{S}^*(\alpha)$ and $\mathscr{C}_\alpha \subseteq \mathscr{C}(\alpha)$. For $\beta < 1$, $\alpha \in \mathbb{R}$, a function $f \in \mathscr{A}$ belongs to the class $\mathscr{R}(\alpha,\beta)$ if it satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right) > \beta.$$

Clearly, $\mathscr{R}(0,\beta) = \mathscr{S}^*(\beta)$. For $\beta \geq -\alpha/2$, Li and Owa [6] proved that $\mathscr{R}(\alpha,\beta) \subset \mathscr{S}^*$.

A function $f \in \mathscr{S}$ is *k-uniformly convex* ($k \geq 0$), if f maps every circular arc γ contained in $\mathbb{D}$ with center ζ, $|\zeta| \leq k$, onto a convex arc. The class of k-uniformly convex functions is denoted by $k-\mathscr{UCV}$. Goodman [4] introduced the class $\mathscr{UCV} := 1-\mathscr{UCV}$ while the class $k-\mathscr{UCV}$ was introduced by Kanas and Wisniowska [8]. They showed that $f \in k-\mathscr{UCV}$ [8, Theorem 2.2, p. 329] (see also [3] for details) if and only if f satisfies the inequality

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right).$$

This analytic characterization is used to obtain the following sufficient condition for a function to be k-uniformly convex.

Theorem 1.1 ([8, Theorem 3.3, p. 334])**.** *If $f(z) = z+\sum_{n=2}^\infty a_n z^n$ satisfies the inequality $\sum_{n=2}^\infty n(n-1)|a_n| \leq 1/(k+2)$ $(k \geq 0)$, then $f \in k-\mathscr{UCV}$. The bound $1/(k+2)$ cannot be replaced by a larger number.*

The above result extended Goodman's [4, Theorem 6] case of $k = 1$ for functions to be k-uniformly convex. In the special case $k = 0$, Theorem 1.1 shows that the constant is $1/2$ for functions f to be convex.

A function $f \in \mathscr{A}$ is *parabolic starlike of order α* if

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-2\alpha+\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right).$$

A sufficient coefficient inequality condition for functions to be parabolic starlike is given in the following result.

Theorem 1.2 ([2, Theorem 3.1, p. 564])**.** *If $f(z) = z+\sum_{n=2}^\infty a_n z^n$ satisfies the inequality $\sum_{n=2}^\infty (n-1)|a_n| \leq (1-\alpha)/(2-\alpha)$, then f is parabolic starlike of order α. The bound $(1-\alpha)/(2-\alpha)$ cannot be replaced by a larger number.*

Further to Theorems 1.1 and 1.2, the present paper determines the largest bound β for analytic functions $f(z) = z+\sum_{n=2}^\infty a_n z^n$ satisfying the inequality $\sum_{n=2}^\infty n(n-1)|a_n| \leq \beta$ to be either starlike or convex of some positive order. In Section 3, a similar problem is investigated for functions f satisfying the coefficient inequality $\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-\beta\right)|a_n|\le 1-\beta$. Section 4 looks at starlike and convex functions of positive order with negative coefficients. In these classes, the largest value is obtained that bounds each coefficient inequality of the form $\sum na_n$, $\sum n(n-1)a_n$, $\sum(n-1)a_n$ and $\sum n^2a_n$. The final section of the paper applies the results obtained to hypergeometric functions.

The following necessary and sufficient conditions for functions to belong to certain subclasses of starlike and convex functions will be used in the sequel.

Theorem 1.3 ([9, Theorem 2, p. 961], [11, Theorem 1 & Corollary, p. 110])**.**

1. *If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ satisfies the inequality*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\le 1-\alpha, \tag{2}$$

then $f\in\mathcal{S}_\alpha^*$. If $a_n\le 0$, then condition (2) is necessary for $f\in\mathcal{S}^*(\alpha)$.

2. *Similarly, if f satisfies the inequality*

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\le 1-\alpha, \tag{3}$$

then $f\in\mathcal{C}_\alpha$. If $a_n\le 0$, then condition (3) is necessary for $f\in\mathcal{C}(\alpha)$.

Theorem 1.3 (1) was proved independently by Merkes, Robertson, and Scott [9, Theorem 2, p. 961] in 1962, and by Silverman [11, Theorem 1, p. 110] in 1975. Theorem 1.3 (2) follows by an application of Alexandar's result, and it was proved in [11, Corollary, p. 110].

2. Sufficient coefficient estimates for starlikeness and convexity

The following theorem provides a sufficient coefficient inequality for functions to be in the classes $\mathcal{C}_\alpha$ or $\mathcal{S}_\alpha^*$.

Theorem 2.1. *Let $\alpha\in[0,1)$, and $f\in\mathcal{A}$ given by (1) satisfy the inequality*

$$\sum_{n=2}^{\infty}n(n-1)|a_n|\le\beta<1. \tag{4}$$

(1) *The function f belongs to the class $\mathcal{C}_\alpha$ if $\beta\le(1-\alpha)/(2-\alpha)$. The bound $(1-\alpha)/(2-\alpha)$ is sharp.*

(2) *The function f belongs to the class $\mathcal{S}_\alpha^*$ if $\beta\le 2(1-\alpha)/(2-\alpha)$. The bound $2(1-\alpha)/(2-\alpha)$ is sharp.*

Proof. (1) Let f satisfy inequality (4) with $\beta \le (1-\alpha)/(2-\alpha)$. Since

$$n-\alpha \le (2-\alpha)(n-1) \tag{5}$$

for $n \ge 2$, the inequality (4) leads to

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le (2-\alpha)\sum_{n=2}^{\infty} n(n-1)|a_n| \le (2-\alpha)\beta \le 1-\alpha.$$

Thus, it follows from Theorem 1.3 (2) that $f \in \mathscr{C}_\alpha$. The function $f_0 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_0(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2$$

satisfies the hypothesis of Theorem 1.3 and therefore $f_0 \in \mathscr{C}_\alpha$. This function f_0 shows that the bound for β is sharp.

(2) Now, let f satisfy inequality (4) with $\beta \le 2(1-\alpha)/(2-\alpha)$. When $n \ge 2$, inequality (5) leads to

$$(n-\alpha) \le \frac{n(n-\alpha)}{2} \le \frac{(2-\alpha)n(n-1)}{2},$$

and hence

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le \frac{(2-\alpha)}{2}\sum_{n=2}^{\infty} n(n-1)|a_n| \le (1-\alpha).$$

By Theorem 1.3(1), $f \in \mathscr{S}^*_\alpha$. The function

$$f_0(z) = z - \frac{1-\alpha}{2-\alpha} z^2 \in \mathscr{S}^*_\alpha$$

shows that the result is sharp. □

Corollary 2.2. [8, Theorem 3.3, p. 334] *If $f \in \mathscr{A}$ given by (1) satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \le \frac{1}{k+2},$$

then $f \in k-\mathscr{UCV}$. Further, the bound $1/(k+2)$ is sharp.

Proof. By Theorem 2.1 (1), it follows that $f \in \mathscr{C}_{k/(k+1)}$, and hence

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{k+1}. \tag{6}$$

Inequality (6) yields

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \frac{k}{k+1} = 1 - \frac{1}{k+1} < 1 - \left|\frac{zf''(z)}{f'(z)}\right| < 1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right),$$

and hence $f \in k-\mathscr{UCV}$. □

It is evident that Alexander's relation holds between the classes $\mathscr{C}_\alpha$ and $\mathscr{S}^*_\alpha$. Thus $f \in \mathscr{C}_\alpha$ if and only if $zf' \in \mathscr{S}^*_\alpha$, and Theorem 2.1 (1) readily yields the following result.

Corollary 2.3. *Let $\alpha \in [0,1)$. If $f \in \mathcal{A}$ is given by (1) and*

$$\sum_{n=2}^{\infty}(n-1)|a_n| \le \frac{1-\alpha}{2-\alpha},$$

then $f \in \mathcal{S}^*_\alpha$. Further, the bound $(1-\alpha)/(2-\alpha)$ is sharp.

The corollary above can also be deduced from Theorem 1.3 (1) and the inequality $n-\alpha \le (2-\alpha)(n-1)$, $n \ge 2$.

Remark 2.4. Theorem 1.2 for the class of parabolic starlike functions of order ρ was obtained by Ali [2, Theorem 3.1, p. 564] by using a two-variable characterization of a corresponding class of uniformly convex functions.

Theorem 2.5. *Let $\alpha \in [0,1)$ and $f \in \mathcal{A}$ be given by (1).*

(1) *If $\sum_{n=2}^{\infty} n|a_n| \le 1-\alpha$, then $f \in \mathcal{S}^*_\alpha$.*

(2) *If $\sum_{n=2}^{\infty} n^2|a_n| \le 1-\alpha$, then $f \in \mathcal{C}_\alpha$.*

(3) *If $\sum_{n=2}^{\infty} n^2|a_n| \le 4(1-\alpha)/(2-\alpha)$, then $f \in \mathcal{S}^*_\alpha$ and the bound $4(1-\alpha)/(2-\alpha)$ is sharp.*

Proof. The first two parts follow from Theorem 1.3 and the simple inequality $n-\alpha < n$. The third follows from Theorem 1.3 (1) and use of the identity $(n-\alpha) \le n^2(2-\alpha)/4 \quad (n \ge 2)$. The result is sharp as demonstrated by the function f_0 given by

$$f_0(z) = z - \frac{1-\alpha}{2-\alpha}z^2. \qquad \square$$

3. The subclass $\mathcal{R}(\alpha,\beta)$

As introduced earlier, the class $\mathcal{R}(\alpha,\beta)$ consists of functions f satisfying the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)\right) > \beta, \quad (\beta < 1,\ \alpha \in \mathbb{R}). \tag{7}$$

The following lemma provides a sufficient coefficient condition for functions f to belong to the class $\mathcal{R}(\alpha,\beta)$.

Lemma 3.1. ([7, Theorem 6, p. 412]) *Let $\beta < 1$, and $\alpha \ge 0$. If $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \le 1-\beta, \tag{8}$$

then $f \in \mathcal{R}(\alpha,\beta)$.

In the special case $\alpha = 0$, Lemma 3.1 reduces to Theorem 1.3 (1). The following theorem provides sufficient coefficient conditions for functions to belong to either $\mathcal{R}(\alpha,\beta)\cap\mathcal{S}^*{}_\eta$ or $\mathcal{R}(\alpha,\beta)\cap\mathcal{C}_\eta$ for an appropriate value η.

Theorem 3.2. *Let $\beta < 1$, $\alpha > 0$, and $f \in \mathcal{A}$ satisfy inequality (8).*

(1) *A function f is in the class $\mathcal{S}^*{}_\eta$ if $\eta \le (2\alpha+\beta)/(2\alpha+1)$. The bound $(2\alpha+\beta)/(2\alpha+1)$ is sharp.*

(2) *A function f is in the class $\mathcal{C}_\eta$ if $\eta \le (\alpha-1+\beta)/\alpha$, and $\beta \ge 0$.*

Proof. (1) If $\eta \le \eta_0 := (2\alpha+\beta)/(2\alpha+1)$, then $\mathcal{S}^*_{\eta_0} \subset \mathcal{S}^*_\eta$. Hence it is enough to prove that $f \in \mathcal{S}^*_{\eta_0}$. The inequality

$$(2\alpha+1)n - 2\alpha \le \alpha n^2 + (1-\alpha)n \quad (n \ge 2,\ \alpha \ge 0)$$

together with inequality (8) show that

$$\begin{aligned}\sum_{n=2}^{\infty}(n-\eta_0)|a_n| &= \sum_{n=2}^{\infty}\frac{(2\alpha+1)n-2\alpha-\beta}{2\alpha+1}|a_n| \\ &\le \sum_{n=2}^{\infty}\frac{\alpha n^2+(1-\alpha)n-\beta}{2\alpha+1}|a_n| \\ &\le \frac{1-\beta}{2\alpha+1} = 1-\eta_0.\end{aligned}$$

It is now evident from Theorem 1.3 (1) that $f \in \mathcal{S}^*{}_{\eta_0}$. The result is sharp for the function $f_0 \in \mathcal{S}^*{}_{\eta_0}$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta}z^2.$$

(2) If $\eta \le \eta_0 := (\alpha-1+\beta)/\alpha$, then $\mathcal{C}_{\eta_0} \subset \mathcal{C}_\eta$. Hence it suffices to show $f \in \mathcal{C}_{\eta_0}$. The inequality

$$\alpha n^2 + (1-\alpha)n - n\beta \le \alpha n^2 + (1-\alpha)n - \beta \quad (n \ge 2, \quad \beta \ge 0)$$

together with inequality (8) yield

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-\eta_0)|a_n| &= \frac{1}{\alpha}\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-n\beta\right)|a_n| \\ &\le \frac{1}{\alpha}\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-\beta\right)|a_n| \\ &\le \frac{1-\beta}{\alpha} = 1-\eta_0.\end{aligned}$$

It follows now from Theorem 1.3(2) that $f \in \mathcal{C}_{\eta_0}$. □

Along similar lines with Theorem 2.1, the following result provides a sufficient coefficient inequality for functions to belong to the class $\mathcal{R}(\alpha,\beta)$.

Theorem 3.3. *Let* $\beta < 1$ *and* $f \in \mathscr{A}$.

(1) *If* f *satisfies* $\sum_{n=2}^{\infty} n(n-1)|a_n| \le 2(1-\beta)/(2\alpha+2-\beta)$, $\alpha \ge 0$, *then* $f \in \mathscr{R}(\alpha,\beta)$. *The bound* $2(1-\beta)/(2\alpha+2-\beta)$ *is sharp.*

(2) *Let* $0 \le \alpha \le 1$ *and* $\eta \in \mathbb{R}$ *be given by*

$$\eta = \begin{cases} 4(1-\beta)/(3\alpha+1), & \alpha+\beta \ge 1, \\ 4(1-\beta)/(2\alpha+2-\beta), & \alpha+\beta \le 1. \end{cases}$$

If f *satisfies* $\sum_{n=2}^{\infty} n^2|a_n| \le \eta$, *then* $f \in \mathscr{R}(\alpha,\beta)$. *The result is sharp for* $\alpha+\beta \le 1$.

Proof.

(1) Let f satisfy $\sum_{n=2}^{\infty} n(n-1)|a_n| \le 2(1-\beta)/(2\alpha+2-\beta)$. Since

$$2\alpha n^2 + 2(1-\alpha)n - 2\beta \le (2\alpha+2-\beta)n(n-1), \quad n \ge 2,$$

it follows that

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \le \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(2\alpha+2-\beta)|a_n| \le 1-\beta.$$

Lemma 3.1 now yields $f \in \mathscr{R}(\alpha,\beta)$. The result is sharp for the function $f_0 \in \mathscr{R}(\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta} z^2.$$

(2) Let $\alpha+\beta \ge 1$ and f satisfy $\sum_{n=2}^{\infty} n^2|a_n| \le 4(1-\beta)/(3\alpha+1)$. In this case, since

$$4\left(\alpha n^2 + (1-\alpha)n - \beta\right) \le (3\alpha+1)n^2 \quad (n \ge 2),$$

it readily follows that

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \le \frac{3\alpha+1}{4}\sum_{n=2}^{\infty} n^2|a_n| \le 1-\beta.$$

Lemma 3.1 shows that $f \in \mathscr{R}(\alpha,\beta)$.

Now, let $\alpha+\beta \le 1$ and the function f satisfy $\sum_{n=2}^{\infty} n^2|a_n| \le 4(1-\beta)/(2\alpha+2-\beta)$. In this case, the inequality

$$4\left(\alpha n^2 + (1-\alpha)n - \beta\right) \le n^2(2\alpha+2-\beta) \quad (n \ge 2)$$

shows that

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \le \frac{1}{4}\sum_{n=2}^{\infty} n^2(2\alpha+2-\beta)|a_n| \le 1-\beta,$$

and hence, Lemma 3.1 implies that $f \in \mathcal{R}(\alpha,\beta)$. The function $f_0 \in \mathcal{R}(\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta} z^2$$

demonstrates sharpness of the result. ☐

4. Functions with negative coefficients

In this section, certain classes of functions with negative coefficients are investigated. The class of functions with negative coefficients, denoted by $\mathcal{T}$, consists of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{9}$$

Denote by $\mathcal{TS}^*(\alpha)$, $\mathcal{TS}^*_\alpha$, $\mathcal{TC}(\alpha)$, and $\mathcal{TC}_\alpha$ the respective subclasses of functions with negative coefficients in $\mathcal{S}^*(\alpha)$, $\mathcal{S}^*_\alpha$, $\mathcal{C}_\alpha$ and $\mathcal{C}(\alpha)$. For starlike and convex functions functions with negative coefficients, Silverman [11] obtained the following result.

Theorem 4.1 ([11, Theorem 2, p. 110], [11, Corollary 2, p. 111])**.** *Let* $\alpha \in [0,1)$, *and* $f \in \mathcal{T}$ *be given by* (9). *Then*

$$f \in \mathcal{TS}^*(\alpha) \Longleftrightarrow f \in \mathcal{TS}^*_\alpha \Longleftrightarrow \sum_{n=2}^{\infty} (n-\alpha) a_n \leq 1-\alpha,$$

and

$$f \in \mathcal{TC}(\alpha) \Longleftrightarrow f \in \mathcal{TC}_\alpha \Longleftrightarrow \sum_{n=2}^{\infty} n(n-\alpha) a_n \leq 1-\alpha.$$

For functions with negative coefficients, the next theorem proves the equivalence between the inequalities $\sum_{n=2}^{\infty} n(n-1)a_n \leq \beta$ and $|f''(z)| < \beta$.

Theorem 4.2. *Let* $\beta > 0$. *If* $f \in \mathcal{T}$ *is given by* (9), *then*

$$|f''(z)| \leq \beta \Longleftrightarrow \sum_{n=2}^{\infty} n(n-1) a_n \leq \beta.$$

Proof. The necessary condition follows by allowing $z \to 1^-$ in

$$|f''(z)| = \left| \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \right| \leq \beta.$$

If f satisfies the coefficient inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq \beta$, then

$$|f''(z)| \leq \sum_{n=2}^{\infty} n(n-1) a_n |z|^{n-2} \leq \sum_{n=2}^{\infty} n(n-1) a_n \leq \beta.$$ ☐

Remark 4.3. It is known that functions $f \in \mathcal{A}$ satisfying the inequality $|f''(z)| \le \beta$ for $0 < \beta \le 1$ are starlike, and if $|f''(z)| \le \beta$ for $0 < \beta \le 1/2$, then $f \in \mathcal{C}$ [13, Theorem 1, p.1861].

Theorem 4.4. *Let* $0 \le \alpha < 1$.

(1) *If* $f \in \mathcal{TC}(\alpha)$, *then* $\sum_{n=2}^{\infty} n a_n \le (1-\alpha)/(2-\alpha)$. *The bound* $(1-\alpha)/(2-\alpha)$ *is sharp.*

(2) *If* $f \in \mathcal{TC}(\alpha)$, *then* $\sum_{n=2}^{\infty} n(n-1) a_n \le 1-\alpha$.

(3) *If* $f \in \mathcal{TC}(\alpha)$, *then* $\sum_{n=2}^{\infty} (n-1) a_n \le (1-\alpha)/2(2-\alpha)$. *The bound* $(1-\alpha)/2(2-\alpha)$ *is sharp.*

(4) *If* $f \in \mathcal{TC}(\alpha)$, *then* $\sum_{n=2}^{\infty} n^2 a_n \le 2(1-\alpha)/(2-\alpha)$. *The bound* $2(1-\alpha)/(2-\alpha)$ *is sharp.*

Proof. The results follow respectively from Theorem 4.1 and the simple inequalities $2-\alpha \le n-\alpha$, $n-1 \le n-\alpha$, $2(2-\alpha)(n-1) \le n(n-\alpha)$, and $n^2(2-\alpha) \le 2n(n-\alpha)$ for all $n \ge 2$. Sharpness of the result are demonstrated by the function f_0 given by

$$f_0(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2. \qquad \square$$

$\square$

The Alexander's relation between $\mathcal{TC}(\alpha)$ and $\mathcal{TS}^*(\alpha)$ readily yields the following corollary.

Corollary 4.5. *Let* $0 \le \alpha < 1$.

(1) *If* $f \in \mathcal{TS}^*(\alpha)$, *then* $\sum_{n=2}^{\infty} a_n \le (1-\alpha)/(2-\alpha)$. *The bound* $(1-\alpha)/(2-\alpha)$ *is sharp.*

(2) *If* $f \in \mathcal{TS}^*(\alpha)$, *then* $\sum_{n=2}^{\infty} (n-1) a_n \le 1-\alpha$.

(3) *If* $f \in \mathcal{TS}^*(\alpha)$, *then* $\sum_{n=2}^{\infty} n a_n \le 2(1-\alpha)/(2-\alpha)$. *The bound* $2(1-\alpha)/(2-\alpha)$ *is sharp.*

We conclude this section with the investigation on functions with negative coefficients in the class $\mathcal{R}(\alpha,\beta)$. The class of all such functions is denoted in the sequel by $\mathcal{TR}(\alpha,\beta)$. The following lemma is needed.

Lemma 4.6. [7, Theorem 8, p.414] *Let* $\beta < 1$, $\alpha \ge 0$, *and* $f \in \mathcal{T}$. *Then,*

$$f \in \mathcal{TR}(\alpha,\beta) \Longleftrightarrow \sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right) a_n \le 1-\beta.$$

Corollary 4.7. *Let* $\beta < 1$, $\alpha > 0$ *and* $f \in \mathcal{TR}(\alpha,\beta)$.

(1) *The function* $f \in \mathcal{TS}^*{}_\eta$ *provided* $\eta \le (2\alpha+\beta)/(2\alpha+1)$, *and the bound* $(2\alpha+\beta)/(2\alpha+1)$ *is sharp.*

(2) *The function* $f \in \mathcal{TC}_\eta$ *provided* $\eta \le (\alpha-1+\beta)/\alpha$, *and* $\beta \ge 0$.

Proof. The result follows from Lemma 4.6 and Theorem 3.2. $\square$

The next result shows that $\mathcal{TC}\big((2\alpha+3\beta-2)/(2\alpha+\beta)\big) \subset \mathcal{TR}(\alpha,\beta)$ for $0 \le \beta < 1$, $\alpha \in \mathbb{R}$.

Theorem 4.8. *Let* $0 \le \beta < 1$, *and* $\alpha \ge 0$. *If* $\eta \ge (2\alpha+3\beta-2)/(2\alpha+\beta)$, *then* $\mathcal{TC}(\eta) \subseteq \mathcal{TR}(\alpha,\beta)$.

Proof. For $\eta_0 \le \eta$, $\mathcal{TC}(\eta) \subset \mathcal{TC}(\eta_0)$ and therefore it is sufficient to prove $\mathcal{TC}(\eta_0) \subseteq \mathcal{TR}(\alpha,\beta)$ where $\eta_0 = (2\alpha+3\beta-2)/(2\alpha+\beta)$. For $n \ge 2$, the inequality

$$2\alpha n^2 + 2(1-\alpha)n - 2\beta \le n\big((2\alpha+\beta)n - (2\alpha+3\beta-2)\big)$$

holds, and Theorem 4.1 yields

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-\beta\right)a_n &\le \frac{1}{2}\sum_{n=2}^{\infty} n\big((2\alpha+\beta)n-(2\alpha+3\beta-2)\big)a_n\\ &= \frac{2\alpha+\beta}{2}\sum_{n=2}^{\infty} n(n-\eta_0)a_n\\ &\le \frac{2\alpha+\beta}{2}(1-\eta_0)\\ &= 1-\beta.\end{aligned}$$

It is now evident from Lemma 4.6 that $f \in \mathcal{TR}(\alpha,\beta)$. □

Theorem 4.9. *Let* $\beta < 1$, *and* $f \in \mathcal{TR}(\alpha,\beta)$.

(1) *Then* $\sum_{n=2}^{\infty} n(n-1)a_n < (1-\beta)/\alpha$ *when* $\alpha > 0$.

(2) $\sum_{n=2}^{\infty}(n-1)a_n \le \eta$ *where* $\eta = (1-\beta)/(1-\alpha)$, $\beta \le 3\alpha+1$, *and* $0 \le \alpha < 1$.

(3) *For* $0 \le \alpha \le 1$, *and*

$$\eta = \begin{cases} (1-\beta)/\alpha, & \beta \le 2(1-\alpha), \alpha > 0\\ 4(1-\beta)/(2\alpha+2-\beta), & \beta \ge 2(1-\alpha), \beta \ge 0, \alpha > 1/2,\end{cases}$$

then $\sum_{n=2}^{\infty} n^2 a_n \le \eta$. *The result for* $\beta \ge 2(1-\alpha)$ *is sharp.*

(4) $\sum_{n=2}^{\infty} n a_n \le 2(1-\beta)/(2\alpha+2-\beta)$, $\alpha, \beta \ge 0$. *The result is sharp.*

Proof. The equivalence in Lemma 4.6 between $f \in \mathcal{TR}(\alpha,\beta)$ and

$$\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-\beta\right)a_n \le 1-\beta$$

is used throughout the proof of this theorem.

(1) Since

$$\alpha n(n-1) \le \alpha n^2 + (1-\alpha)n - \beta, \quad n \ge 2,$$

it readily follows that

$$\sum_{n=2}^{\infty} n(n-1)a_n < \sum_{n=2}^{\infty}\frac{\alpha n^2+(1-\alpha)n-\beta}{\alpha}a_n \le \frac{1-\beta}{\alpha}.$$

(2) If $\beta \le 3\alpha + 1$, then

$$(n-1)(1-\alpha) \le \alpha n(n-1) + n - \beta, \quad n \ge 2,$$

and use of this inequality shows that

$$\sum_{n=2}^{\infty}(n-1)a_n \le \sum_{n=2}^{\infty}\frac{\alpha n^2 + (1-\alpha)n - \beta}{1-\alpha}a_n \le \frac{1-\beta}{1-\alpha}.$$

(3) If $\beta \le 2(1-\alpha)$, the inequality

$$\alpha n^2 \le \alpha n^2 + 2(1-\alpha) - \beta \le \alpha n^2 + n(1-\alpha) - \beta$$

shows that

$$\sum_{n=2}^{\infty} n^2 a_n \le \sum_{n=2}^{\infty}\frac{\alpha n^2 + (1-\alpha)n - \beta}{\alpha}a_n \le \frac{1-\beta}{\alpha}.$$

In the case $\beta \ge 2(1-\alpha)$, the inequality

$$n^2(2\alpha + 2 - \beta) \le 4(\alpha n^2 + (1-\alpha)n - \beta), \quad n \ge 2,$$

readily gives

$$\sum_{n=2}^{\infty} n^2 a_n \le \sum_{n=2}^{\infty}\frac{4(\alpha n^2 + (1-\alpha)n - \beta)}{2\alpha + 2 - \beta}a_n \le \frac{4(1-\beta)}{2\alpha + 2 - \beta}.$$

(4) For $\alpha, \beta \ge 0$, the inequality

$$(2\alpha + 2 - \beta)n \le 2\big(\alpha n^2 + (1-\alpha)n - \beta\big)$$

shows that

$$\sum_{n=2}^{\infty} n a_n \le \sum_{n=2}^{\infty}\frac{2\big(\alpha n^2 + (1-\alpha)n - \beta\big)}{2\alpha + 2 - \beta}a_n \le \frac{2(1-\beta)}{2\alpha + 2 - \beta}.$$

The sharpness can be seen by considering the function f_0 given by

$$f(z) = z - \frac{1-\beta}{2\alpha + 2 - \beta}z^2 \in \mathcal{TR}(\alpha, \beta). \qquad \square$$

5. Applications to Gaussian hypergeometric functions

For $a, b, c \in \mathbb{C}$ with $c \ne 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is defined by

$$F(a, b; c; z) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots,$$

where $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (n = 0, 1, 2, \ldots).$$

The series converges absolutely in $\mathbb{D}$. It also converges on $|z| = 1$ when $\operatorname{Re}(c-a-b) > 0$. For $\operatorname{Re}(c-a-b) > 0$, the value of the hypergeometric function $F(a,b;c;z)$ at $z = 1$ is related to the Gamma function by the Gauss summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0,-1,-2,\ldots). \tag{10}$$

By making use of Theorem 1.3, Silverman [12] determined conditions on a,b,c so that the function $zF(a,b;c;z)$ belongs to certain subclasses of starlike and convex functions. In the following theorem, conditions on the parameters a,b,c are determined so that the function $zF(a,b;c;z)$ belongs to the class $\mathscr{R}(\alpha,\beta)$. With regard to the other classes investigated in this paper, similar results could also be obtained. The proof follows directly by applying appropriate theorems from the previous sections, the Gauss summation formula for the Gaussian hypergeometric functions, and certain straight forward manipulations; the method of proof is similar to those of Silverman [12], and Kim and Ponnusamy [5]. The following Gauss summation formula for the Gaussian hypergeometric functions is required.

Lemma 4.10. [1, Lemma 10, p.169] *Let a, b, $c > 0$.*

(1) *If $c > a+b+1$,*

$$\sum_{n=1}^{\infty} n\frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{ab}{c-a-b-1}F(a,b;c;1).$$

(2) *If $c > a+b+2$,*

$$\sum_{n=1}^{\infty} n^2\frac{(a)_n(b)_n}{(c)_n(1)_n} = \left(\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right)F(a,b;c;1).$$

Theorem 4.11. *Let $a,b \in \mathbb{C}$ and $c \in \mathbb{R}$ satisfy $c > |a|+|b|+2$. If either*

$$F(|a|,|b|;c;1)\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{2|ab|}{c-|a|-|b|-1}\right) \leq \frac{2(1-\beta)}{2\alpha+2-\beta} \tag{11}$$

for $\alpha \geq 0$, $\beta < 1$, or

$$F(|a|,|b|;c;1)\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{3|ab|}{c-|a|-|b|-1} + 1\right) \leq \frac{6-5\beta+2\alpha}{2\alpha+2-\beta} \tag{12}$$

for $1-\alpha \geq \beta$, $\alpha \in [0,1]$, then the function $zF(a,b;c;z) \in \mathscr{R}(\alpha,\beta)$. In the case $b = \overline{a}$, the range of c in either case can be improved to $c > \max\{0, 2(1+\operatorname{Re} a)\}$.

Proof. For$\alpha \geq 0$, $\beta < 1$, it follows from the fact $|(a)_n| \leq (|a|)_n$ and Lemma 4.10 that

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| &\leq \sum_{n=2}^{\infty} n(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= F(|a|,|b|;c;1)\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{2|ab|}{c-|a|-|b|-1}\right)\end{aligned}$$

$$\leq \frac{2(1-\beta)}{2\alpha+2-\beta},$$

and Theorem 3.3 (1) shows that $zF(a,b;c;z) \in \mathscr{R}(\alpha,\beta)$.

For $1-\alpha \geq \beta$, $\alpha \in [0,1]$, it follows from Lemma 4.10 that

$$\begin{aligned}\sum_{n=2}^{\infty} n^2 \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| &\leq \sum_{n=2}^{\infty} n^2 \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= F(|a|,|b|;c;1)\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{3|ab|}{c-|a|-|b|-1} + 1\right) - 1 \\ &\leq \frac{4(1-\beta)}{2\alpha+2-\beta}.\end{aligned}$$

The result follows from Theorem 3.3 (2). The proof for the case $b = \overline{a}$ is similar. □

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References

[1] O. P. Ahuja, *Planar harmonic convolution operators generated by hypergeometric functions*, Integral Transforms Spec. Funct., **18** (2007), 165–177.

[2] R. M. Ali, *Starlikeness associated with parabolic regions*, Int. J. Math. Math. Sci., (2005), 561–570.

[3] R. M. Ali, V. Ravichandran, *Uniformly convex and uniformly starlike functions*, Math. Newsletter, Ramanujan Math. Soc., **21**(2011), 16–30.

[4] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., **56**(1991), 87–92.

[5] Y. C. Kim and S. Ponnusamy, *Sufficiency for Gaussian hypergeometric functions to be uniformly convex*, Int. J. Math. Math. Sci., **22**(1999), 765–773.

[6] J.-L. Li and S. Owa, *Sufficient conditions for starlikeness*, Indian J. Pure Appl. Math., **33**(2002), 313–318.

[7] M.-S. Liu, Y.-C. Zhu and H. M. Srivastava, *Properties and characteristics of certain subclasses of starlike functions of order β*, Math. Comput. Modelling, **48** (2008), 402–419.

[8] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., **105** (1999), 327–336.

[9] E. P. Merkes, M. S. Robertson and W. T. Scott, *On products of starlike functions*, Proc. Amer. Math. Soc., **13**(1962), 960–964.

[10] S. Ponnusamy and F. Rønning, *Starlikeness properties for convolutions involving hypergeometric series*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **52** (1998), 141–155.

[11] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**(1975), 109–116.

[12] H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., **172** (1993), 574–581.

[13] V. Singh, *On some problems of Mocanu type*, Indian J. Pure Appl. Math., **32** (2001), 1859–1867.

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