NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS WHOSE POLES ARE LOCALLY UNIFORMLY DISCRETE

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Abstract. Let $h$ be a positive number, and let $a(z)$ be a function holomorphic and zero-free on a domain $D$. Let $F$ be a family of meromorphic functions on $D$ such that for every $f \in F$, $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f''(z)| \leq h$. Suppose that each pair of functions $f$ and $g$ in $F$ have the same poles. Then $F$ is normal on $D$.

1. Introduction

A family $F$ of meromorphic functions on a plane domain $D \subset \mathbb{C}$ is said to be normal on $D$ in the sense of Montel if each sequence $\{f_n\} \in F$ contains a subsequence which converges spherically uniformly on each compact subset of $D$. See [4], [7], [9].

For two functions $f$ and $g$ meromorphic on $D$ in $\mathbb{C}$, and two complex numbers or meromorphic functions $a$ and $b$, we write $f(z) = a(z) \Rightarrow g(z) = b(z)$ if $g(z) = b(z)$ whenever $f(z) = a(z)$, and write $f(z) = a(z) \Leftrightarrow g(z) = b(z)$ if $f(z) = a(z)$ if and only if $g(z) = b(z)$. When $a$ is a complex value and $f(z) = a \Leftrightarrow g(z) = a$, we also say that $f$ and $g$ share the value $a$ or $a$ is a shared value of $f$ and $g$. For families of meromorphic functions, the connection between normality and shared values has been studied frequently following Schwick’s initial paper [8].

The starting point of this paper is the following result.

Theorem 1.1 ([1] Theorem 4). Let $h$ be a positive number, $k \geq 2$ be an integer, and let $a(z)$ be a function which is holomorphic and zero-free on $D$. Then the family $F = \{f\}$ of holomorphic functions on $D$ such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f^{(k)}(z)| \leq h$ is normal on $D$.

We remark that with the same notations, the family $F = \{f\}$ of meromorphic functions on $D$ such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f^{(k)}(z)| \leq h$ is not normal in general (at least for $k = 2$) even if $a(z)$ is a nonzero constant. This is shown by Example 1 in [2] where
it is proved that such a family $F$ is normal if the poles of each function in $F$ have sufficiently large multiplicities. We give here another condition on poles that enables the family $F$ to be normal. In the sequel, we say that the poles of functions in $F$ are locally uniformly discrete on $D$, if for each point $z_0 \in D$, there exists $\delta = \delta(z_0) > 0$ such that each function $f \in F$ has at most one pole (ignoring multiplicity) in the disk $\Delta(z_0, \delta) = \{z : |z - z_0| < \delta\} \subset D$.

**Theorem 1.2.** Let $h$ be a positive number and $a(z)$ be a function which is holomorphic and zero-free on $D$. Then the family $F = \{f\}$ of meromorphic functions on $D$ such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f''(z)| \leq h$ is normal on $D$, provided that the poles of functions in $F$ are locally uniformly discrete on $D$.

Notice that if each pair of functions $f$ and $g$ in $F$ have the same poles on $D$, then the poles of functions in $F$ are locally uniformly discrete in $D$. So we have the following corollary to Theorem 1.2.

**Theorem 1.3.** Let $h$ be a positive number and $a(z)$ be a function which is holomorphic and zero-free on $D$. Then the family $F = \{f\}$ of meromorphic functions on $D$ such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f''(z)| \leq h$ is normal on $D$, provided that each pair of functions $f$ and $g$ in $F$ have the same poles on $D$.

### 2. Some Lemmas

In order to prove our theorems, we require the following results. We assume the standard notation of value distribution theory, as presented and used in [4]. The first lemma is a special case of [6, Lemma 2].

**Lemma 2.1.** Let $F$ be a family of functions meromorphic on $D$. Suppose that there exists $A \geq 1$ such that $f(z) = 0 \Rightarrow |f'(z)| \leq A$ for each $f \in F$. If $F$ is not normal at some point $z_0 \in D$, there exist points $z_n \in D$ with $z_n \to z_0$, positive numbers $\rho_n \to 0$ and functions $f_n \in F$ such that the sequence $g_n(\zeta) = \rho_{n}^{-1} f_n(z_n + \rho_n \zeta)$ converges spherically locally uniformly on $\mathbb{C}$ to a nonconstant meromorphic function $g$ which is of finite order and satisfies $g^{\sharp}(\zeta) \leq g^{\sharp}(0) = A + 1$.

Here, as usual, $g^{\sharp}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

**Lemma 2.2** ([5]). Let $f$ be a meromorphic function on $\mathbb{C}$ of finite order. Then for each positive integer $k$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r), \ r \to \infty.$$

**Lemma 2.3** ([4] Corollary to Theorem 3.5). Let $f$ be a transcendental meromorphic function, and let $a$ be a non-zero value. Then, for each positive integer $k$, either $f$ or $f^{(k)} - a$ has infinitely many zeros.
Lemma 2.4 ([2] Remark, [1] Lemma 6). Let $g$ be a nonconstant entire function, $k \geq 2$ be an integer, and let $a$ be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$ and $g''(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where $z_0$ is a constant.

Lemma 2.5. Let $g$ be a function meromorphic on $\mathbb{C}$ of finite order and with finitely many poles, and let $a$ be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g''(z) = 0$, then $g$ is a rational function.

Proof. Suppose that $g$ is transcendental, then by Lemma 2.4, $g$ is not entire and by the assumption, $N(r, g) = O(\log r)$. Since $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g''(z) = 0$, the zeros of $g$ are simple and $g(z) = 0 \Rightarrow g''(z) = 0$. It follows that

$$N \left( r, \frac{g''}{g} \right) \leq 2N(r, g) = O(\log r).$$

On the other hand, as $g$ is of finite order, by Lemma 2.2, we have

$$m \left( r, \frac{g''}{g} \right) = O(\log r).$$

Hence

$$T \left( r, \frac{g''}{g} \right) = m \left( r, \frac{g''}{g} \right) + N \left( r, \frac{g''}{g} \right) = O(\log r).$$

This shows that the function

$$R = \frac{g''}{g}$$

is a rational function. Since $g$ is not entire, $R$ is nonconstant and has at least one (double) pole. Thus by the condition $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g''(z) = 0$, we see that the zeros of $g$ and the zeros of $g' - a$ coincide with finitely many exceptions, and that all zeros of $g' - a$ are double with finitely many exceptions. Hence by Lemma 2.3, $g$ has infinitely many zeros, and as $g$ is of finite order,

$$\frac{g' - a}{g^2} = Qe^U$$

for some polynomials $Q(\neq 0)$ and $U$. By (2.2), we have

$$g' = a + Qe^U g^2,$$

so that

$$g'' = (Q' + QU)e^U g^2 + 2Qe^U g' g = (Q' + QU)e^U g^2 + 2Qe^U g(g^2 Qe^U + a)$$

$$= 2Q^2 e^{2U} g^3 + (Q' + QU)e^U g^2 + 2aQe^U g.$$ (2.4)

Thus by (2.1), we have

$$2Q^2 e^{2U} g^3 + (Q' + QU)e^U g + 2aQe^U = R.$$ (2.5)
Hence
\[ g(z) = 0 \Rightarrow 2aQe^U = R(z). \tag{2.6} \]
Now differentiating both sides of (2.5) yields that
\[ 2(Q^2e^{2U}g^2) + [(Q' + QU')e^U] g + (Q' + QU')e^U g' + 2a(Q' + QU')e^U = R', \tag{2.7} \]
so that by \( g(z) = 0 \Rightarrow g'(z) = a \), we get
\[ g(z) = 0 \Rightarrow 3a(Q' + QU')e^U = R'. \tag{2.8} \]
Hence, by (2.6) and (2.8), we see that
\[ g(z) = 0 \Rightarrow \frac{3}{2} \left( \frac{Q'}{Q} + U' \right) = \frac{R'}{R}. \tag{2.9} \]
Since \( g \) has infinitely many zeros, it follows that
\[ \frac{3}{2} \left( \frac{Q'}{Q} + U' \right) \equiv \frac{R'}{R}. \tag{2.10} \]
We claim that (2.10) is impossible. In fact, the residues of the right at the double poles of \( R \) are –2, while the residues of the left are not negative everywhere.

This contradiction shows that \( g \) must be a rational function. This completes the proof of the lemma. \( \square \)

**Lemma 2.6.** Let \( g \) be a nonconstant rational function with at most one pole, and let \( a \) be a nonzero finite value. If \( g(z) = 0 \Rightarrow g'(z) = a \) and \( g'(z) = a \Rightarrow g''(z) = 0 \), then \( g(z) = a(z - z_0) \), where \( z_0 \) is a constant.

**Proof.** By Lemma 2.4, we only have to prove that \( g \) is a polynomial. Suppose not, then by the condition, we may assume \( g \) has only one pole 0 and it has multiplicity \( m \geq 1 \). Hence 0 is a double pole of \( g''/g \). On the other hand, since \( g(z) = 0 \Rightarrow g'(z) = a \) and \( g'(z) = a \Rightarrow g''(z) = 0 \), the zeros of \( g \) are simple and \( g(z) = 0 \Rightarrow g''(z) = 0 \), so that \( g''/g \) has no other poles, and hence \( z^2 g''/g \) is a polynomial. Since
\[ \frac{g''}{g} = \frac{g''}{g'} \cdot \frac{g'}{g} = O \left( \frac{1}{z} \right) \cdot O \left( \frac{1}{z} \right) = O \left( \frac{1}{z^2} \right) \]
as \( z \to \infty \), we then see that \( z^2 g''/g \) is a constant \( c \). Obviously, \( c \neq 0 \).

Now write
\[ g(z) = z^{-m} \sum_{s=0}^{n} a_s z^s = \sum_{s=0}^{n} a_s z^{s-m}, \]
where $a_s$ are constants and $a_0 \neq 0$. Then

$$g'' = \sum_{s=0}^{n} (s-m)(s-m-1)a_sz^{s-m-2}.$$  

Thus by $z^2g'' = cg$, we get

$$\sum_{s=0}^{n} (s-m)(s-m-1)a_sz^s = c \sum_{s=0}^{n} a_sz^s. \quad (2.11)$$

So we can see $(s-m)(s-m-1)a_s = ca_s$, $s = 0, 1, 2, \ldots , n$. It follows from $a_0 \neq 0$ that $c = m(m+1)$, and hence $a_s = 0, s = 1, 2, \ldots , 2m, 2m+2, \ldots , n$.

If $a_{2m+1} = 0$, we have $g(z) = a_0z^{-m}$. Obviously, the zeros of $g'(z) - a$ are all simple, which contradicts $g'(z) = a \Rightarrow g''(z) = 0$.

If $a_{2m+1} \neq 0$, we have $g(z) = a_0z^{-m} + a_{2m+1}z^{m+1}$. Then

$$g'(z) - a = \frac{H(z)}{z^{m+1}},$$

where

$$H(z) = (m+1)a_{2m+1}z^{2m+1} - az^{m+1} - ma_0. \quad (2.12)$$

Since $g'(z) = a \Rightarrow g''(z) = 0$, we see that all zeros of $H(z)$ are multiple. Suppose $\zeta$ is a zero of $H(z)$, then

$$H(\zeta) = (m+1)a_{2m+1}\zeta^{2m+1} - a\zeta^{m+1} - ma_0 = 0 \quad (2.13)$$

and

$$H'(\zeta) = (m+1)(2m+1)a_{2m+1}\zeta^{2m} - (m+1)a\zeta^m = 0. \quad (2.14)$$

We see that $\zeta \neq 0$, and by (2.14),

$$\zeta^m = \frac{a}{(2m+1)a_{2m+1}}.$$  

Then by (2.13),

$$(m+1)a_{2m+1} \left( \frac{a}{(2m+1)a_{2m+1}} \right)^2 \zeta - \frac{a^2}{(2m+1)a_{2m+1}} \zeta - ma_0 = 0,$$

hence we have

$$\zeta = \frac{(2m+1)^2a_0a_{2m+1}}{a^2}.$$  

This shows that $\zeta$ is a unique zero of $H(z)$, then we get

$$H(z) = (m+1)a_{2m+1} \left( z + \frac{(2m+1)^2a_0a_{2m+1}}{a^2} \right)^{2m+1}. \quad (2.15)$$

Comparing the coefficient of $z$ of (2.12) and (2.15), there is a contradiction.

The proof is completed. □
3. Proofs of the main results

Proof of Theorem 1.2. It suffices to show that \( \mathcal{F} \) is normal at every point in \( D \). Suppose that \( \mathcal{F} \) is not normal at some point \( z_0 \in D \). By the condition, there exists \( \delta > 0 \) such that every \( f \) in \( \mathcal{F} \) has at most one pole (simple or multiple) in \( \Delta(z_0, \delta) \) and \( \overline{\Delta(z_0, \delta)} \subset \mathcal{F} \). Then by Lemma 2.1, we can find functions \( f_n \in \mathcal{F} \), points \( z_n \in \Delta(z_0, \delta) \) with \( z_n \to z_0 \), and positive numbers \( \rho_n \to 0 \) such that the sequence of functions \( g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n \zeta) \) converges spherically locally uniformly on \( \mathbb{C} \) to a nonconstant meromorphic function \( g \) of finite order such that \( g''(0) = (M + 1) + 1 = M + 2 \), where \( M = \max \{|a(z)| : z \in \overline{\Delta(z_0, \delta)}\} \).

We claim that

(i) \( g \) has at most one pole;

(ii) \( g(\zeta) = 0 \Rightarrow g'(\zeta) = a(z_0) \); and

(iii) \( g'(\zeta) = a(z_0) \Rightarrow g''(\zeta) = 0 \).

To prove (i), suppose that \( g(\zeta) \) has two distinct poles \( \zeta_1 \) and \( \zeta_2 \). Then by Hurwitz’s theorem, \( g_n \) (for \( n \) sufficiently large) has two distinct poles \( \zeta_{n,1} \) and \( \zeta_{n,2} \) such that \( \zeta_{n,j} \to \zeta_j \), \( (j = 1, 2) \). It follows from \( g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n \zeta) \) that \( f_n \) has two distinct poles \( w_{n,1} = z_n + \rho_n \zeta_{n,1} \) and \( w_{n,2} = z_n + \rho_n \zeta_{n,2} \). However, as both \( w_{n,j} \to z_0 \), this contradicts that every \( f \) in \( \mathcal{F} \) has at most one pole in \( \Delta(z_0, \delta) \). The claim (i) is proved.

To prove (ii), let \( \zeta_0 \) be a zero of \( g \). Then by Hurwitz’s theorem, there exist points \( \zeta_n \to \zeta_0 \), such that \( g_n(\zeta_n) = 0 \) (for \( n \) sufficiently large), and hence \( f_n(z_n + \rho_n \zeta_n) = 0 \). Thus by \( f_n(\zeta) = 0 \Rightarrow f_n'(\zeta) = a(\zeta) \), we have \( g_n'(\zeta_n) = f_n'(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n) \). It follows that \( g'(\zeta_0) = \lim_{n \to \infty} g_n'(\zeta_n) = a(z_0) \). This proves (ii).

Next we prove (iii). Suppose that \( g'(\zeta_0) = a(z_0) \). Then \( g(\zeta_0) \neq \infty \) and hence on some neighborhood \( \Delta(\zeta_0) \) of \( \zeta_0 \), \( g \) and \( g_n \) (with \( n \) sufficiently large) are holomorphic. Thus \( g_n'(\zeta) - a(z_n + \rho_n \zeta) \to g'(\zeta) - a(z_0) \) and \( g_n''(\zeta) \to g''(\zeta) \) on \( \Delta(\zeta_0) \). Hence by \( g'(\zeta) \neq a(z_0) \) (for otherwise, \( g(\zeta) = a(z_0) \zeta + c \) for some constant \( c \) which contradicts that \( g''(0) = M + 2 > |a(z_0)| \)) and Hurwitz’s theorem, there exist points \( \zeta_n \to \zeta_0 \) such that \( g_n'(\zeta_n) - a(z_n + \rho_n \zeta_n) = 0 \) (for \( n \) sufficiently large). Since \( g_n'(\zeta) = f_n'(z_n + \rho_n \zeta_n) \), we get \( f_n'(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n) \), and hence \( |f_n'(z_n + \rho_n \zeta_n)| \leq h \) by the condition \( f'(z) = a(z) \Rightarrow |f''(z)| \leq h \). Since \( g_n''(\zeta) = \rho_n f_n''(z_n + \rho_n \zeta_0) \), it follows that \( |g_n''(\zeta_n)| \leq \rho_n h \), and hence \( g''(\zeta_0) = 0 \) by \( g_n''(\zeta) \to g''(\zeta) \). The claim (iii) is also proved.

Now by Lemma 2.5 and Lemma 2.6, \( g \) must be of the form \( g(\zeta) = a(z_0) \zeta + c \) for some constant \( c \). This contradicts that \( g''(0) = M + 2 > |a(z_0)| \).

Thus \( \mathcal{F} \) is normal at every point in \( D \) and hence on \( D \). \( \square \)
References


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