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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS WHOSE POLES ARE LOCALLY UNIFORMLY DISCRETE

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Abstract. Let h be a positive number, and let $a(z)$ be a function holomorphic and zero-free on a domain D. Let $\mathscr{F}$ be a family of meromorphic functions on D such that for every $f \in \mathscr{F}$, $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f''(z)| \le h$. Suppose that each pair of functions f and g in $\mathscr{F}$ have the same poles. Then $\mathscr{F}$ is normal on D.

1. Introduction

A family $\mathscr{F}$ of meromorphic functions on a plane domain $D \subset \mathbb{C}$ is said to be normal on D in the sense of Montel if each sequence $\{f_n\} \in \mathscr{F}$ contains a subsequence which converges spherically uniformly on each compact subset of D. See [4], [7], [9].

For two functions f and g meromorphic on D in $\mathbb{C}$, and two complex numbers or meromorphic functions a and b, we write $f(z) = a(z) \Rightarrow g(z) = b(z)$ if $g(z) = b(z)$ whenever $f(z) = a(z)$, and write $f(z) = a(z) \Leftrightarrow g(z) = b(z)$ if $f(z) = a(z)$ if and only if $g(z) = b(z)$. When a is a complex value and $f(z) = a \Leftrightarrow g(z) = a$, we also say that f and g share the value a or a is a shared value of f and g. For families of meromorphic functions, the connection between normality and shared values has been studied frequently following Schwick's initial paper [8].

The starting point of this paper is the following result.

Theorem 1.1 ([1] Theorem 4)**.** *Let h be a positive number, $k \ge 2$ be an integer, and let $a(z)$ be a function which is holomorphic and zero-free on D. Then the family $\mathscr{F} = \{f\}$ of holomorphic functions on D such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f^{(k)}(z)| \le h$ is normal on D.*

We remark that with the same notations, the family $\mathscr{F} = \{f\}$ of meromorphic functions on D such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f^{(k)}(z)| \le h$ is not normal in general (at least for $k = 2$) even if $a(z)$ is a nonzero constant. This is shown by Example 1 in [2] where

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it is proved that such a family $\mathscr{F}$ is normal if the poles of each function in $\mathscr{F}$ have sufficiently large multiplicities. We give here another condition on poles that enables the family $\mathscr{F}$ to be normal. In the sequel, we say that the poles of functions in $\mathscr{F}$ are locally uniformly discrete on D, if for each point $z_0 \in D$, there exists $\delta = \delta(z_0) > 0$ such that each function $f \in \mathscr{F}$ has at most one pole (ignoring multiplicity) in the disk $\Delta(z_0, \delta) = \{z : |z - z_0| < \delta\} \subset D$.

Theorem 1.2. *Let h be a positive number and $a(z)$ be a function which is holomorphic and zero-free on D. Then the family $\mathscr{F} = \{f\}$ of meromorphic functions on D such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f''(z)| \le h$ is normal on D, provided that the poles of functions in $\mathscr{F}$ are locally uniformly discrete on D.*

Notice that if each pair of functions f and g in $\mathscr{F}$ have the same poles (or share the value ∞) on D, then the poles of functions in $\mathscr{F}$ are locally uniformly discrete in D. So we have the following corollary to Theorem 1.2.

Theorem 1.3. *Let h be a positive number and $a(z)$ be a function which is holomorphic and zero-free on D. Then the family $\mathscr{F} = \{f\}$ of meromorphic functions on D such that $f(z) = 0 \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow |f''(z)| \le h$ is normal on D, provided that each pair of functions f and g in $\mathscr{F}$ have the same poles on D.*

2. Some lammas

In order to prove our theorems, we require the following results. We assume the standard notation of value distribution theory, as presented and used in [4]. The first lemma is a special case of [6, Lemma 2].

Lemma 2.1. *Let $\mathscr{F}$ be a family of functions meromorphic on D. Suppose that there exists $A \ge 1$ such that $f(z) = 0 \Rightarrow |f'(z)| \le A$ for each $f \in \mathscr{F}$. If $\mathscr{F}$ is not normal at some point $z_0 \in D$, there exist points $z_n \in D$ with $z_n \to z_0$, positive numbers $\rho_n \to 0$ and functions $f_n \in \mathscr{F}$ such that the sequence $g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n \zeta)$ converges spherically locally uniformly on $\mathbb{C}$ to a nonconstant meromorphic function g which is of finite order and satisfies $g^\sharp(\zeta) \le g^\sharp(0) = A + 1$.*

Here, as usual, $g^\sharp(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative.

Lemma 2.2 ([5])**.** *Let f be a meromorphic function on $\mathbb{C}$ of finite order. Then for each positive integer k,*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r), \quad r \to \infty.$$

Lemma 2.3 ([4] Corollary to Theorem 3.5)**.** *Let f be a transcendental meromorphic function, and let a be a non-zero value. Then, for each positive integer k, either f or $f^{(k)} - a$ has infinitely many zeros.*

Lemma 2.4 ([2] Remark, [1] Lemma 6). *Let g be a nonconstant entire function, $k \geq 2$ be an integer, and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

Lemma 2.5. *Let g be a function meromorphic on $\mathbb{C}$ of finite order and with finitely many poles, and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g''(z) = 0$, then g is a rational function.*

Proof. Suppose that g is transcendental, then by Lemma 2.4, g is not entire and by the assumption, $N(r, g) = O(\log r)$. Since $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g''(z) = 0$, the zeros of g are simple and $g(z) = 0 \Rightarrow g''(z) = 0$. It follows that

$$N\left(r, \frac{g''}{g}\right) \leq 2N(r, g) = O(\log r).$$

On the other hand, as g is of finite order, by Lemma 2.2, we have

$$m\left(r, \frac{g''}{g}\right) = O(\log r).$$

Hence

$$T\left(r, \frac{g''}{g}\right) = m\left(r, \frac{g''}{g}\right) + N\left(r, \frac{g''}{g}\right) = O(\log r).$$

This shows that the function

$$R = \frac{g''}{g} \tag{2.1}$$

is a rational function. Since g is not entire, R is nonconstant and has at least one (double) pole. Thus by the condition $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g''(z) = 0$, we see that the zeros of g and the zeros of $g' - a$ coincide with finitely many exceptions, and that all zeros of $g' - a$ are double with finitely many exceptions. Hence by Lemma 2.3, g has infinitely many zeros, and as g is of finite order,

$$\frac{g' - a}{g^2} = Qe^U \tag{2.2}$$

for some polynomials $Q(\not\equiv 0)$ and U. By (2.2), we have

$$g' = a + Qe^U g^2, \tag{2.3}$$

so that

$$\begin{aligned} g'' &= (Q' + QU')e^U g^2 + 2Qe^U g g' = (Q' + QU')e^U g^2 + 2Qe^U g(g^2 Qe^U + a) \\ &= 2Q^2 e^{2U} g^3 + (Q' + QU')e^U g^2 + 2aQe^U g. \end{aligned} \tag{2.4}$$

Thus by (2.1), we have

$$2Q^2 e^{2U} g^2 + (Q' + QU')e^U g + 2aQe^U = R. \tag{2.5}$$

Hence

$$g(z)=0\Rightarrow 2aQe^{U}=R(z). \tag{2.6}$$

Now differentiating the both sides of (2.5) yields that

$$2\left(Q^2e^{2U}g^2\right)'+\left[(Q'+QU')e^{U}\right]'g+(Q'+QU')e^{U}g'+2a(Q'+QU')e^{U}=R', \tag{2.7}$$

so that by $g(z)=0\Rightarrow g'(z)=a$, we get

$$g(z)=0\Rightarrow 3a(Q'+QU')e^{U}=R'. \tag{2.8}$$

Hence, by (2.6) and (2.8), we see that

$$g(z)=0\Rightarrow \frac{3}{2}\left(\frac{Q'}{Q}+U'\right)=\frac{R'}{R}. \tag{2.9}$$

Since g has infinitely many zeros, it follows that

$$\frac{3}{2}\left(\frac{Q'}{Q}+U'\right)\equiv\frac{R'}{R}. \tag{2.10}$$

We claim that (2.10) is impossible. In fact, the residues of the right at the double poles of R are -2, while the residues of the left are not negative everywhere.

This contradiction shows that g must be a rational function. This completes the proof of the lemma. □

Lemma 2.6. *Let g be a nonconstant rational function with at most one pole, and let a be a nonzero finite value. If $g(z)=0\Rightarrow g'(z)=a$ and $g'(z)=a\Rightarrow g''(z)=0$, then $g(z)=a(z-z_0)$, where z_0 is a constant.*

Proof. By Lemma 2.4, we only have to prove that g is a polynomial. Suppose not, then by the condition, we may assume g has only one pole 0 and it has multiplicity $m\geq 1$. Hence 0 is a double pole of g''/g. On the other hand, since $g(z)=0\Rightarrow g'(z)=a$ and $g'(z)=a\Rightarrow g''(z)=0$, the zeros of g are simple and $g(z)=0\Rightarrow g''(z)=0$, so that g''/g has no other poles, and hence z^2g''/g is a polynomial. Since

$$\frac{g''}{g}=\frac{g''}{g'}\cdot\frac{g'}{g}=O\left(\frac{1}{z}\right)\cdot O\left(\frac{1}{z}\right)=O\left(\frac{1}{z^2}\right)$$

as $z\to\infty$, we then see that z^2g''/g is a constant c. Obviously, $c\neq 0$.

Now write

$$g(z)=z^{-m}\sum_{s=0}^{n}a_sz^{s}=\sum_{s=0}^{n}a_sz^{s-m},$$

where a_s are constants and $a_0 \neq 0$. Then

$$g'' = \sum_{s=0}^{n} (s-m)(s-m-1)a_s z^{s-m-2}.$$

Thus by $z^2 g'' = cg$, we get

$$\sum_{s=0}^{n} (s-m)(s-m-1)a_s z^s = c\sum_{s=0}^{n} a_s z^s. \tag{2.11}$$

So we can see $(s-m)(s-m-1)a_s = ca_s$, $s = 0,1,2,\ldots,n$. It follows from $a_0 \neq 0$ that $c = m(m+1)$, and hence $a_s = 0$, $s = 1,2,\ldots,2m,2m+2,\ldots,n$.

If $a_{2m+1} = 0$, we have $g(z) = a_0 z^{-m}$. Obviously, the zeros of $g'(z) - a$ are all simple, which contradicts $g'(z) = a \Rightarrow g''(z) = 0$.

If $a_{2m+1} \neq 0$, we have $g(z) = a_0 z^{-m} + a_{2m+1} z^{m+1}$. Then

$$g'(z) - a = \frac{H(z)}{z^{m+1}},$$

where

$$H(z) = (m+1)a_{2m+1} z^{2m+1} - az^{m+1} - ma_0. \tag{2.12}$$

Since $g'(z) = a \Rightarrow g''(z) = 0$, we see that all zeros of $H(z)$ are multiple. Suppose ζ is a zero of $H(z)$, then

$$H(\zeta) = (m+1)a_{2m+1}\zeta^{2m+1} - a\zeta^{m+1} - ma_0 = 0 \tag{2.13}$$

and

$$H'(\zeta) = (m+1)(2m+1)a_{2m+1}\zeta^{2m} - (m+1)a\zeta^{m} = 0. \tag{2.14}$$

We see that $\zeta \neq 0$, and by (2.14),

$$\zeta^m = \frac{a}{(2m+1)a_{2m+1}}.$$

Then by (2.13),

$$(m+1)a_{2m+1}\left(\frac{a}{(2m+1)a_{2m+1}}\right)^2 \zeta - \frac{a^2}{(2m+1)a_{2m+1}}\zeta - ma_0 = 0,$$

hence we have

$$\zeta = -\frac{(2m+1)^2 a_0 a_{2m+1}}{a^2}.$$

This shows that ζ is a unique zero of $H(z)$, then we get

$$H(z) = (m+1)a_{2m+1}\left(z + \frac{(2m+1)^2 a_0 a_{2m+1}}{a^2}\right)^{2m+1}. \tag{2.15}$$

Comparing the coefficient of z of (2.12) and (2.15), there is a contradiction.

The proof is completed. □

3. Proofs of the main results

Proof of Theorem 1.2. It suffices to show that $\mathscr{F}$ is normal at every point in D. Suppose that $\mathscr{F}$ is not normal at some point $z_0 \in D$. By the condition, there exists $\delta > 0$ such that every f in $\mathscr{F}$ has at most one pole (simple or multiple) in $\Delta(z_0, \delta)$ and $\overline{\Delta}(z_0, \delta) \subset D$. Then by Lemma 2.1, we can find functions $f_n \in \mathscr{F}$, points $z_n \in \Delta(z_0, \delta)$ with $z_n \to z_0$, and positive numbers $\rho_n \to 0$ such that the sequence of functions $g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n \zeta)$ converges spherically locally uniformly on $\mathbb{C}$ to a nonconstant meromorphic function g of finite order such that $g^\sharp(0) = (M+1)+1 = M+2$, where $M = \max\{|a(z)| : z \in \overline{\Delta}(z_0, \delta)\}$.

We claim that

(i) g has at most one pole;

(ii) $g(\zeta) = 0 \Rightarrow g'(\zeta) = a(z_0)$; and

(iii) $g'(\zeta) = a(z_0) \Rightarrow g''(\zeta) = 0$.

To prove (i), suppose that $g(\zeta)$ has two distinct poles ζ_1 and ζ_2. Then by Hurwitz's theorem, g_n (for n sufficiently large) has two distinct poles $\zeta_{n,1}$ and $\zeta_{n,2}$ such that $\zeta_{n,j} \to \zeta_j, (j = 1, 2)$. It follows from $g_n(\zeta) = \rho_n^{-1} f_n(z_n + \rho_n \zeta)$ that f_n has has two distinct poles $w_{n,1} = z_n + \rho_n \zeta_{n,1}$ and $w_{n,2} = z_n + \rho_n \zeta_{n,2}$. However, as both $w_{n,j} \to z_0$, this contradicts that every f in $\mathscr{F}$ has at most one pole in $\Delta(z_0, \delta)$. The claim (i) is proved.

To prove (ii), let ζ_0 be a zero of g. Then by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$, such that $g_n(\zeta_n) = 0$ (for n sufficiently large), and hence $f_n(z_n + \rho_n \zeta_n) = 0$. Thus by $f_n(\zeta) = 0 \Rightarrow f_n'(\zeta) = a(\zeta)$, we have $g_n'(\zeta_n) = f_n'(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n)$. It follows that $g'(\zeta_0) = \lim_{n\to\infty} g_n'(\zeta_n) = a(z_0)$. This proves (ii).

Next we prove (iii). Suppose that $g'(\zeta_0) = a(z_0)$. Then $g(\zeta_0) \neq \infty$ and hence on some neighborhood $\Delta(\zeta_0)$ of ζ_0, g and g_n (with n sufficiently large) are holomorphic. Thus $g_n'(\zeta) - a(z_n + \rho_n \zeta) \to g'(\zeta) - a(z_0)$ and $g_n''(\zeta) \to g''(\zeta)$ on $\Delta(\zeta_0)$. Hence by $g'(\zeta) \not\equiv a(z_0)$ (for otherwise, $g(\zeta) = a(z_0)\zeta + c$ for some constant c which contradicts that $g^\sharp(0) = M + 2 > |a(z_0)|$) and Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $g_n'(\zeta_n) - a(z_n + \rho_n \zeta_n) = 0$ (for n sufficiently large). Since $g_n'(\zeta) = f_n'(z_n + \rho_n \zeta)$, we get $f_n'(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n)$, and hence $|f_n''(z_n + \rho_n \zeta_n)| \le h$ by the condition $f'(z) = a(z) \Rightarrow |f''(z)| \le h$. Since $g_n''(\zeta) = \rho_n f_n''(z_n + \rho_n \zeta)$, it follows that $|g_n''(\zeta_n)| \le \rho_n h$, and hence $g''(\zeta_0) = 0$ by $g_n''(\zeta) \to g''(\zeta)$. The claim (iii) is also proved.

Now by Lemma 2.5 and Lemma 2.6, g must be of the form $g(\zeta) = a(z_0)\zeta + c$ for some constant c. This contradicts that $g^\sharp(0) = M + 2 > |a(z_0)|$.

Thus $\mathscr{F}$ is normal at every point in D and hence on D. □

References

[1] J. M. Chang, M. L. Fang and L. Zalcman, *Normal families of holomorphic functions*, J. Math. Illinois, **48**(2004), 319–337.

[2] J. M. Chang, *A note on normality of meromorphic functions*, Proc. Japan Acad., **83**, Ser. A(2007), 60–62.

[3] J. Clunie and W. K. Hayman, *The spherical derivative of integral and meromorphic functions*, Comment. Math. Helv., **40**(1966), 117–148.

[4] W. K. Hayman, Meromorphic Functions, Clarendon Press, Oxford,1964.

[5] K. L. Hiong, *Sur les fonctions holomorphes dont les dérivées admettent une valeur exceptionnelle,* Ann. Sci. École Norm. Sup. (3), **72**(1955), 165–197.

[6] X. C. Pang and L. Zalcman, *Normal families and shared values*, Bull. London Math. Soc., **32**(2000), 325–331.

[7] J. L. Schiff, Normal Families, Springer-Verlag, New York, 1993.

[8] W. Schwick, *Sharing values and normality*, Arch Math., **59**(1993), 50–54.

[9] L. Yang, Value Distribution Theory, Spring-Verlag, Berlin, 1993.

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