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EXISTENCE OF SOLUTIONS OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper we prove the existence of mild and strong solutions of semilinear integrodifferential equations in Banach spaces with nonlocal initial conditions. We prove the existence theorems by using Schaefer's fixed point theorem.

1. Introduction

In this paper we consider semilinear integrodifferential equations with nonlocal conditions

$$\begin{aligned} &y'(t) = A(t,y)y + f\left(t, y, \int_0^t K(t,s)F(s,y(s))ds\right), \quad t \in J = [0,b], \\ &y(0) + g(y) = y_0, \end{aligned} \tag{1}$$

where $A(t,y) : E \to E$, $f : J \times E \times E \to E$ and $K : J \times J \to R$ are continuous functions, $g : C(J,E) \to E$, $y_0 \in E$ and E is a real Banach space with the norm $\|\cdot\|$. We prove the existence of mild and strong solutions for the above problem by using Schaefer's fixed point theorem.

Nonlinear differential equations with classical initial conditions have been studied by many authors [1, 5]. The nonlocal conditions which is a generalization of the classical initial condition was considered by Byszewski [4]. Several papers have been devoted to studying the existence of solutions for differential equations with nonlocal conditions [2, 3, 4]. In this paper we investigate the mild and strong solutions of the semilinear integrodifferential equations with nonlocal conditions in Banach spaces.

2. Preliminaries and Basic Hypotheses

Let $C(J,E)$ be the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \sup\{\|y(t)\|/t \in J\}.$$

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and let $B(E)$ denote the Banach space of bounded linear operators from E into E with norm

$$\|N\|_{B(E)} = \sup\{\|Ny\|/\|y\| = 1\}.$$

The following lemma and fixed point theorem are useful for proving the existence theorems.

Lemma 1. ([6], p.36) *Suppose that* Φ_1, $\Phi_2 \in C(J,R)$, $\Phi_3 \in L^1(J,R)$, $\Phi_3(t) \geq 0$ *a.e. on* J *and* $\Phi_1(t) \leq \Phi_2(t) + \int_0^t \Phi_3(s)\Phi_1(s)ds$. *Then*

$$\Phi_1(t) \leq \Phi_2(t) + \int_0^t \Phi_3(s)\Phi_2(s)\exp\left(\int_s^t \Phi_3(\tau)d\tau\right) ds.$$

Fixed point theorem: [7]

Let E be a Banach space and let $N : E \to E$ be a continuous compact map. If the set $\Omega = \{y \in E : \lambda y = N(y)$ for some $\lambda > 1\}$ is bounded, then N has a fixed point.

Now, let us assume the following hypotheses:

(H1) $A : J \times E \to B(E)$ is a continuous function such that $\forall r > 0$, $\exists r_1 = r_1(r) > 0$ such that $\|v\| \leq r \Rightarrow \|A(t,v)\|_{B(E)} \leq r_1$, $\forall\, t \in J$, $\forall\, v \in E$.

(H2) $f : J \times E \times E \to E$, $(t,u,v) \to f(t,u,v)$ is a continuous function.

(H3) There exists a constant $L > 0$ such that $\|g(y)\| \leq L$ for some $y \in E$.

(H4) $K : J \times J \to R$ is a continuous function such that $\|K(t,s)\| \leq L_1$ for some constant L_1 and for some t, $s \in J$.

(H5) $F : J \times E \to E$ is a continuous function.

(H6) $\|F(t,x)_1\| \leq q(t)H(\|x(t)\|)$ for almost all $t \in J$ and all $x \in E$, where $q \in L^1(J,R_+)$ and $H : R_+ \to (0,\infty)$ is continuous and non-decreasing.

(H7) $\|f(t,y,z)\| \leq p(t)\psi(\|y\| + \|z\|)$ for almost all $t \in J$ and all y, $z \in E$, where $p \in L^1(J,R_+)$ and $\psi : R_+ \to (0,\infty)$ is continuous and non-decreasing with

$$M\int_0^b p(s)ds < \int_c^\infty \frac{du}{\psi(u)},$$

where $c = M\|y_0\| + ML$ and $M = \sup\{\|U_y(t,s)\|_{B(E)}/(t,s) \in J \times J\}$.

(H8) For each bounded $B \subset C(J,E)$, $y \in B$ and $t \in J$ the set $\{U_y(t,0)y_0 - U_y(t,0)$ $g(y) + \int_0^t U_y(t,s)f(s,y(s),z(s))ds\}$, where $z(s) = \int_0^t K(t,s)F(s,y(s))ds$, is relatively compact.

(H9) $\int_0^t M_1(s)ds < \int_c^\infty \frac{ds}{M\psi(s)+H(s)}$, where $M_1(t) = \max\{p(t), L_1q(t)\}$.

Remark. From (H1) for any fixed $u \in C(J,E)$ there exists a unique function $U_n : J \times J \to B(E)$ defined and continuous on $J \times J$ such that

$$U_n(t,s) = I + \int_S^t A_u(w)U_u(w,s)dw,$$

where I stands for the identity operator on E and $A_u(t) = A(t,u(t))$. $U_n(t,s)$ is the evolution operator of A.

Definition 1. A continuous solution $y(t)$ of the integral equation

$$y(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s)f(s,y(s),z(s))ds$$

is called a *mild solution* of (1).

Definition 2. A function y is said to be a *strong solution* of (1) on J if, y is differentiable almost everywhere on J,

$$\frac{dy}{dt} \in (L^1[0,b],E),$$
$$y' = A(t,y)y + f(t,y,z) \text{ a.e. on } J \text{ and } y(0) + g(y) = y_0.$$

3. Existence Theorems

3.1. Mild solutions

Theorem 1. *Let $g : C(J,E) \to E$ be a continuous function. Assume that hypotheses* (H1)-(H8) *are satisfied. Then problem* (1) *has at least one mild solution on J.*

Proof. Consider the map $N : C(J,E) \to C(J,E)$ defined by

$$(Ny)(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s)f(s,y(s),z(s))ds,$$

where $t \in J$.

Step 1: $U_u(t,s)$ is continuous with respect to u; that is,

$$\|u_n - u^*\|_\infty \to 0 \Rightarrow$$
$$\|U_{u_n} - U_{u^*}\|_\infty = \sup_{(t,s)\in J\times J} \|U_{u_n}(t,s) - U_{u^*}(t,s)\|_{B(E)} \to 0, \text{ as } n \to \infty.$$

Let $\|u_n - u^*\|_\infty \to 0$. Then there exists $r > 0$ such that $\|u_n\|_\infty$, $\|u^*\|_\infty \le r$. If $s \le t$, then

$$\begin{aligned}
\|U_{u_n} - U_{u^*}\|_\infty &= \sup_{(t,s)\in J\times J} \|U_{u_n}(t,s) - U_{u^*}(t,s)\|_{B(E)} \\
&\le \sup_{(t,s)\in J\times J} \int_s^t \|U_{u_n}(w,s)\|_{B(E)} \|[A_{u_n}(w) - A_{u^*}(w)]\|_{B(E)} dw \\
&\quad + \sup_{(t,s)\in J\times J} \int_s^t \|A_{u^*}(w)\|_{B(E)} \|[U_{u_n}(w,s) - U_{u^*}(w,s)]\|_{B(E)} dw \\
&\le \int_s^t M \|[A_{u_n}(w) - A_{u^*}(w)]\|_{B(E)} dw \\
&\quad + \int_s^t \|A_{u^*}\|_{B(E)} \|[U_{u_n}(w,s) - U_{u^*}(w,s)]\|_{B(E)} dw.
\end{aligned}$$

By Lemma 1,

$$\begin{aligned}
\|U_{u_n} - U_{u^*}\|_\infty &\le M \int_s^t \|[A_{u_n}(w) - A_{u^*}(w)]\|_{B(E)} dw \\
&\quad + M \int_s^t \|A_{u^*}(w)\|_{B(E)} \left(\int_s^t \|A_{u_n}(\tau) - A_{u^*}(\tau)\|_{B(E)} d\tau \right) \\
&\quad \times \left(\exp \int_w^t \|A_{u^*}(z)\|_{B(E)} dz \right) dw \\
&\le M \int_s^t \|A_{u_n} - A_{u^*}\|_\infty dw \\
&\quad + M \int_s^t \|A_{u^*}\|_\infty \left(\int_s^t \|A_{u_n} - A_{u^*}\|_\infty d\tau \right) \\
&\quad \times \exp \left(\int_w^t \|A_{u^*}\|_\infty dz \right) dw \\
&\le bM \|A_{u_n} - A_{u^*}\|_\infty + M \int_s^t \{ \|A_{u^*}\|_\infty b \|A_{u_n} - A_{u^*}\|_\infty \\
&\quad \times \exp(b\|A_{u^*}\|_\infty) \} dw \\
&\le bM \|A_{u_n} - A_{u^*}\|_\infty + Mb^2 \|A_{u^*}\|_\infty \|A_{u_n} - A_{u^*}\|_\infty \exp(b\|A_{u^*}\|_\infty) \\
&\le \|A_{u_n} - A_{u^*}\|_\infty bM[1 + br_1 \exp(br_1)] \quad \text{(by (H1))}.
\end{aligned}$$

Now, $u \in C(J, E)$ implies $A_u \in C(J, B(E))$. Also, $\|u_n - u^*\|_\infty \to 0$ implies $\|A_{u_n} - A_{u^*}\|_\infty = \max\{\|A_{u_n}(t) - A_{u^*}(t)\|_{B(E)}/t \in J\} \to 0$ as $n \to \infty$. Therefore, $\|U_{u_n} - U_{u^*}\|_\infty \to 0$ as $n \to \infty$. Thus $\|u_n - u^*\|_\infty \to 0 \Rightarrow \|U_{u_n} - U_{u^*}\|_\infty \to 0$ as $n \to \infty$. Therefore, $U_n(t, s)$ is continuous with respect to u.

Step 2. N maps bounded sets into relatively compact sets i.e., N is a compact map. Let $B_r = \{y \in C(J, E)/\|y\|_\infty \le r\}$. Clearly B_r is a bounded set in $C(J, E)$. For each $t \in J$ we have

$$(Ny)(t) = U_y(t, 0)y_0 - U_y(t, 0)g(y) + \int_0^t U_y(t, s)f(s, y(s), z(s))ds.$$

Therefore, for each $t \in J$ we have

$$\begin{aligned}
\|Ny\| &\le \|U_y(t, 0)y_0\| + \|U_y(t, 0)g(y)\| + \int_0^t \|U_y(t, s)f(s, y(s), z(s))\| ds \\
&\le \|U_y(t, 0)\|_{B(E)} \|y_0\| + \|U_y(t, 0)\|_{B(E)} \|g(y)\| \\
&\quad + \int_0^t \|U_y(t, s)\|_{B(E)} \|f(s, y(s), z(s))\| ds \\
&\le M\|y_0\| + M\|g(y)\| + \int_0^t M\|f(s, y(s), z(s))\| ds \\
&\le M\|y_0\| + ML + M \int_0^t p(s)\psi(\|y\| + \|z\|) ds.
\end{aligned}$$

Consider, $\|y\|_\infty = \sup\{\|y(t)\|/t \in J\} \leq r$.
Therefore, $\|y\| \leq \sup_{y\in[0,r]}\{\|y(t)\|\}$,

$$\begin{aligned}\|z\| &= \left\| \int_0^t K(t,s)F(s,y(s))ds \right\| \\ &\leq L_1 \int_0^t q(s)H(\|y(s)\|)ds \\ &\leq L_1 \left\{ \sup_{y\in[0,r]} H(y) \right\} \int_0^t q(s)ds.\end{aligned}$$

Therefore,

$$\|y\| + \|z\| \leq \sup_{y\in[0,r]} \|y(t)\| + L_1 \left\{ \sup_{y\in[0,r]} H(y) \right\} \int_0^t q(s)ds.$$

$$\text{Let } \mu(t) = \sup_{y\in[0,r]} \|y(t)\| + L_1 \left\{ \sup_{y\in[0,r]} H(y) \right\} \int_0^t q(s)ds.$$

Then $\|y\| + \|z\| \leq \mu(t)$.
Since ψ is non-decreasing,

$$\|y\| + \|z\| \leq \mu(t) \Rightarrow \psi(\|y\| + \|z\|) \leq \psi(\mu(t)).$$

Therefore, $\|Ny\| \leq M\|y_0\| + ML + M\int_0^t p(s)\psi(\mu(s))ds$
and

$$\|Ny\|_\infty \leq M\|y_0\| + ML + M \sup_{t\in J} \left(\int_0^t p(s)ds \right) \max_{y\in B} \left(\sup_{y\in[0,r]} \psi(y) \right).$$

Now, let $t_1,\, t_2 \in J$, $t_1 < t_2$ and $y \in B_r$.
Then

$$\begin{aligned}&\|(Ny)(t_2) - (Ny)(t_1)\| \\ &\leq \|\{U_y(t_2,0) - U_y(t_1,0)\}y_0\| + \|\{U_y(t_2,0) - U_y(t_1,0)\}g(y)\| \\ &\quad + \left\| \int_0^{t_2} U_y(t_2,s)f(s,y(s),z(s))ds - \int_0^{t_1} U_y(t_1,s)f(s,y(s),z(s))ds \right\| \\ &\leq \|\{U_y(t_2,0) - U_y(t_1,0)\}y_0\| + \|\{U_y(t_2,0) - U_y(t_1,0)\}g(y)\| \\ &\quad + \left\| \int_0^{t_2} U_y(t_2,s)f(s,y(s),z(s))ds - \int_0^{t_2} U_y(t_1,s)f(s,y(s),z(s))ds \right. \\ &\quad \left. + \int_0^{t_2} U_y(t_1,s)f(s,y(s),z(s))ds - \int_0^{t_1} U_y(t_1,s)f(s,y(s),z(s))ds \right\| \\ &\leq \|U_y(t_2,0) - U_y(t_1,0)\|_{B(E)}\|y_0\| + \|U_y(t_2,0) - U_y(t_1,0)\|_{B(E)}\|g(y)\|\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^{t_2} U_y(t_2,s)f(s,y(s),z(s))ds - \int_0^{t_2} U_y(t_1,s)f(s,y(s),z(s))ds\right\| \\
&+\left\|\int_{t_1}^{t_2} U_y(t_1,s)f(s,y(s),z(s))ds\right\| \\
\le\ & \|U_y(t_2,0)-U_y(t_1,0)\|_{B(E)}\|y_0\| + \|U_y(t_2,0)-U_y(t_1,0)\|_{B(E)}\|L \\
&+\left\|\int_0^{t_2} \{U_y(t_2,s)-U_y(t_1,s)\}p(s)\psi(\|y\|+\|z\|)ds\right\| \\
&+M\int_{t_1}^{t_2} p(s)\psi(\|y(s)\|+\|z(s)\|)ds.
\end{aligned}
$$

The string of inequalities is bounded by $k(t_2-t_1)$ for some $k>0$.
Hence, $N(B_r)$ is an equicontinuous family of functions. Therefore, by Ascoli-Arzela theorem, $N(B_r)$ is relatively compact. Therefore, N maps bounded sets into relatively compact sets, i.e., N is a compact map. We have,

$$(Ny)(t) = U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s)f(s,y(s),z(s))ds.$$

Since U_y, g, f are all continuous functions, N is also continuous. Therefore, N is a continuous compact map.

Step 3. The set $\Psi=\{y\in C(J,E)/\lambda y = N(y), \lambda>1\}$ is bounded.

Let $\Omega=\{y\in X/\lambda y = N(y)$ for some $\lambda>1\}$, where X is a Banach space and $N: X\to X$ is a continuous compact map.

Let $y\in\Omega$.
Then $\lambda y = N(y)$ for some $\lambda>1$.

$$
\begin{aligned}
\lambda y(t) &= (Ny)(t) \\
&= U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s)f(s,y(s),z(s))ds, \quad t\in J,
\end{aligned}
$$

and therefore

$$y(t) = \lambda^{-1}U_y(t,0)y_0 - \lambda^{-1}U_y(t,0)g(y) + \lambda^{-1}\int_0^t U_y(t,s)f(s,y(s),z(s))ds, \quad t\in J.$$

Now $\lambda>1$ implies $\lambda^{-1}\le 1$.

Therefore, $$y(t) \le U_y(t,0)y_0 - U_y(t,0)g(y) + \int_0^t U_y(t,s)f(s,y(s),z(s))ds, \quad t\in J.$$

Hence
$$
\begin{aligned}
\|y(t)\| &\le \|U_y(t,0)y_0\| + \|U_y(t,0)g(y)\| + \int_0^t \|U_y(t,s)f(s,y(s),z(s))\|ds \\
&\le M\|y_0\| + ML + M\int_0^t p(s)\psi(\|y(s)\|+\|z(s)\|)ds \text{ for } t\in J.
\end{aligned}
$$

Let us take the right-hand side of the above inequality as $v(t)$. Then

$$v(t) = M\|y_0\| + ML + M\int_0^t p(s)\psi(\|y(s)\| + \|z(s)\|)ds.$$

Now, $v(0) = M\|y_0\| + ML$. Thus we have,

$$v(0) = M\|y_0\| + ML \text{ and } \quad \|y(t)\| \le v(t), \quad t \in J.$$

Since ψ is non-decreasing,

$$\|y(t)\| \le v(t) \Rightarrow \psi(y(t)) \le \psi(v(t)), \quad t \in J.$$

Now,

$$v(t) = M\|y_0\| + ML + M\int_0^t p(s)\psi(\|y(s)\| + \|z(s)\|)ds.$$

Therefore,

$$\begin{aligned} v'(t) &= Mp(t)\psi(\|y(t)\| + \|z(t)\|) \\ &\le Mp(t)\psi\left(v(t) + L_1\int_0^t q(s)H(v(s))ds\right). \end{aligned}$$

Let $u(t) = v(t) + L_1\int_0^t q(s)H(v(s))ds$.

Then

$$\begin{aligned} & v(t) \le u(t) \text{ and} \\ & u(0) = v(0) = M\|y_0\| + ML \\ & u(0) = v'(t) + L_1 q(t)H(v(t)) \\ & \quad \le Mp(t)\psi(u(t)) + L_1 q(t)H(v(t)) \\ & \quad \le Mp(t)\psi(u(t)) + L_1 q(t)H(v(t)) \\ & \quad \le M_1(t)\{M\psi(u(t)) + H(u(t))\}, \end{aligned}$$

where $M_1(t) = \max\{p(t), L_1 q(t)\}$.

$$u'(t)/M\psi(u(t)) + H(u(t)) \le M_1(t).$$

Intregrating, we have

$$\begin{aligned} \int_{u(0)}^{u(t)} \frac{ds}{M\psi(s) + H(s)} &\le \int_0^t M_1(s)ds < \int_0^b M_1(s)ds \\ &\le \int_{u(0)}^{\infty} \frac{ds}{M\psi(s) + H(s)}. \end{aligned}$$

This inequality implies that there exists a constant d such that $u(t) \le d$, $t \in J$. But $v(t) \le u(t)$. Therefore, $v(t) \le d$ for $t \in J$. Also, $\|y(t)\| \le v(t)$ for $t \in J$. Hence

$\|y(t)\| \leq d$ for $t \in J$. Therefore, $\|y\|_\infty \leq d$, where d depends on the functions M_1, ψ and H. Now, $y \in \Omega$ and $\|y\|_\infty \leq d \Rightarrow \Omega$ is bounded. Set $X = C(J, E)$. Then Ψ is bounded.

Now, $C(J, E)$ is a Banach space and $N : C(J, E) \to C(J, E)$ is a continuous compact map. Also, the set

$$\Psi = \{y \in C(J, E) : \lambda y = N(y) \text{ for } \lambda > 1\} \text{ is bounded.}$$

Hence by Schaefer's fixed point theorem, N has a fixed point which is a mild solution of (1).

3.2. Strong solutions

Theorem 2. *Let E be a reflexive Banach space. Let $g : C(J, E) \to E$ be a continuous function. Assume that hypotheses* (H1)-(H9) *are satisfied. Then* (1) *has a strong solution.*

Proof. Since all the assumptions of Theorem 1 are satisfied, (1) has a mild solution. Let $y(t)$ satisfy (1). Let $g_1(s) = U_y(t, s)y(s)$. This $g_1(s)$ is differentiable on J,

and
$$\begin{aligned} dg_1/ds &= [(d/ds)\{U_y(t, s)\}]y(s) + U_y(t, s)y'(s) \\ &= U_y(t, s)f(s, y(s), z(s)). \end{aligned}$$

Since $f \in L^1(J; E)$, $U_y(t, s)f(s, y(s), z(s))$ is integrable. Integrating from 0 to t, we get,

$$\int_0^t \frac{dg_1}{ds} ds = \int_0^t U_y(t, s)f(s, y(s), z(s))ds,$$

$$U_y(t, t)y(t) - U_y(t, 0)y(0) = \int_0^t U_y(t, s)f(s, y(s), z(s))ds.$$

Thus
$$y(t) = U_y(t, 0)y_0 - U_y(t, 0)g(y) + \int_0^t U_y(t, s)f(s, y(s), z(s))ds$$

which is the mild solution of (1).

Now, we will show that this mild solution is a strong solution of (1).

Since f is differentialbe a.e. and $f' \in L^1(J, E)$, y is differentiable a.e. on J and $y' \in L^1(J, E)$. Now,

$$\begin{aligned} \|y(t + h) - y(t)\| &\leq \|\{U_y(t + h, 0) - U_y(t, 0)\}y_0\| \\ &\quad + \|\{U_y(t + h, 0) - U_y(t, 0)\}g(y)\| \\ &\quad + \int_0^t \|\{U_y(t + h, s) - U_y(t, s)\}f(s, y(s), z(s))\|ds \\ &\quad + \int_t^{t+h} \|U_y(t + h, s)f(s, y(s), z(s))\|ds \\ &\leq \|U_y(t + h, 0) - U_y(t, 0)\| \, \|y_0\| \\ &\quad + \|U_y(t + h, 0) - U_y(t, 0)\| \, \|g(y)\| \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|\{U_y(t+h,s)-U_y(t,s)\}\|\,\|f(s,y(s),z(s))\|ds\\
&+\int_t^{t+h} \|U_y(t+h,s)\|\,\|f(s,y(s),z(s))\|ds,\\
\|y(t+h)-y(t)\| &\le r_1Mh\|y_0\|+r_1Mh\|g(y)\|+r_1Mh\int_0^t \|f(s,y(s),z(s))\|ds\\
&+\int_t^{t+h} M\|f(s,y(s),z(s))\|ds\\
&\le r_1Mh\|y_0\|+r_1Mh\|g(y)\|\\
&+r_1Mh\int_0^t p(s)\psi(\|y\|+\|z\|)ds+M\int_0^h p(s+t)\psi(\|y\|+\|z\|)ds\\
&\le hK_1.
\end{aligned}
$$

Thus $\|y(t+h)-y(t)\|\le hK_1$, where K_1 is constant. Therefore, y is Lipschitz continuous.

The Lipschitz continuity of y combined with the continuity of f imply that $t\to f(t,y(t),z(t))$ is Lipschitz continuous.

Since E is reflexive and f is Lipschitz continuous, f is differentiable a.e. on J and $f'\in L^1(J;E)$.

Hence, y is differentiable a.e. on J and $y'\in L^1(J;E)$. Also, $y(t)$ satisfies (1).

Therefore, $y(t)$ is a strong solution of (1) on J.

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