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CONVERGENT AND DIVERGENT SOLUTIONS OF A DISCRETE NONAUTONOMOUS LOTKA-VOLTERRA MODEL

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Abstract. In this paper, a discrete nonautonomous m-species Lotka-Volterra system is investigated. By using fixed point theorems, a set of simple and easily verifiable conditions are given for the existence of convergent or divergent positive solutions.

1. Introduction

We consider the system of difference equations

$$v(n+1) = v(n)\exp\{r(n) + B(n)v(n)\}\,,\ n = 0, 1, 2, \ldots, \tag{1}$$

where $v(n)$ is a column vector $(v_1(n), \ldots, v_m(n))^\dagger$, $r(n)$ is a column vector $(r_1(n), \ldots, r_m(n))^\dagger$ and $B(n) = (b_{kl}(n))_{m\times m}$ is a square m by m matrix for each $n \in \{0, 1, 2, ...\}$. In case $m = 1$, our system is reduced to a scalar difference equation which can be used to describe certain growth models. In the general case, our equation can be used to describe growth models for several species under competition, or arm race models.

Similar models have been studied by many people. For example, Wang et al. in [6] studied the global stability of discrete population models, Xu et al. in [7] investigated a discrete periodic two-species Lotka-Volterra predator-prey model with time delays by using Gaines and Mawhin's continuation theorem of coincidence degree theory, and Li and Ratfoul in [8] studied a classification scheme for the eventually positive solutions of a class of two-dimensional Volterra nonlinear difference equations. Motivated by these and other works [1-13], in the present paper, we discuss the existence of certain solutions of system (1).

Throughout this paper we will assume that the vector sequence $\{r(n)\}_{n=0}^\infty$ and the matrix sequence $\{B(n)\}_{n=0}^\infty$ are nonnegative and bounded, and

$$A_k = \sum_{i=0}^{\infty} r_k(i) < +\infty,\ k = 1, \ldots, m. \tag{2}$$

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We will also be interested in solutions of (1) that originates from positive initial distributions. More specifically, a real vector v is said to be positive (nonnegative) and denoted by $v > 0$ (respectively $v \geq 0$) if its components are positive (respectively nonnegative). Let $v(0) > 0$, then in view of (1), we may determine $v(1)$, $v(2)$, ... in a unique manner. Such a sequence $\{v(n)\}_{n=0}^{\infty}$ is said to be a solution of (1) originated from a positive distribution.

2. Main Results

Note that a solution $\{v(n)\}_{n=0}^{\infty}$ originated from a positive distribution is a positive vector sequence, that is, $v(n) > 0$ for each $n \in \{0, 1, ...\}$. Let

$$u_k(n) = \ln v_k(n), \quad k = 1, \dots, m. \tag{3}$$

Then from (1), we see that

$$\Delta u(n) = r(n) + B(n) \exp(u(n)), \tag{4}$$

where $\exp(u_1, \dots, u_m)^\dagger$ denotes the vector $(\exp u_1, \dots, \exp u_m)^\dagger$. We call the sequence $\{u(n)\}$ the positive sequence associated with the solution $\{v(n)\}$. In view of (3), once the properties of positive solutions of (4) can be obtained, then we may infer from (3) the corresponding properties of solutions of (1) originated from positive initial distributions.

To this end, we first note that if $\{u(n)\}$ is a positive solution of (4), then $\Delta u(n) \geq 0$ for $n \geq 0$ so that each component sequence $\{u_k(n)\}$ is positive and nondecresing. Thus

$$\lim_{n\to\infty} u_k(n) \leq +\infty, \ k = 1, \dots, m,$$

so that each component sequence may be divergent or convergent.

First of all, it is easy to find conditions for every component sequence to diverge.

Theorem 1. *Suppose* (2) *holds and*

$$B_k = \sum_{i=0}^{\infty} \sum_{l=1}^{m} b_{kl}(i) = +\infty, \ k = 1, \dots, m$$

hold. If $\{u(n)\}$ *is a positive solution* (4), *then*

$$\lim_{n\to\infty} u_k(n) = +\infty, \ k = 1, \dots, m.$$

Proof. Let $\{u(n)\}$ be a positive solution of (4). Then summing (4), we see that

$$\begin{aligned} u_k(n) &= u_k(0) + \sum_{i=0}^{n-1} r_k(i) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{kl}(i) \exp(u_l(i)) \\ &\geq \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{kl}(i) \exp(u_l(i)) \geq \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{kl}(i) \exp(u_l(0)) \\ &\geq L \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{kl}(i), \end{aligned}$$

where $L = \min_{1\le l\le m}\{\exp(u_l(0)\}$. Therefore, our proof is completed by taking limits on both sides as n tends to $+\infty$.

Next, we turn to conditions for each component sequence to be convergent.

Teorem 2. *Suppose* (2) *holds. If* $\{u(n)\}$ *is a positive solution of* (4) *such that*

$$\lim_{n\to\infty} u_k(n) = \alpha_k < +\infty,\ \ k = 1,\dots,m \tag{5}$$

then

$$B_k = \sum_{i=0}^{\infty}\sum_{l=1}^{m} b_{kl}(i) < +\infty,\ \ k = 1,\dots,m. \tag{6}$$

Conversely, if (6) *holds, then* (4) *has an eventually positive solution* $\{u(n)\}$ *that satisfies* (5).

Proof. Suppose that $\{u(n)\}$ is a positive solution of (4) such that $\lim_{n\to\infty} u_k(n) = \alpha_k < +\infty$ for $k = 1,\dots,m$. Then, there exists an integer $N > 0$ and positive constants $c_1,\dots,c_m$ such that $c_k \le u_k \le \alpha_k$ for $n \ge N$ and $k = 1,\dots,m$. In view of (4), for $n \ge N$,

$$u_k(n) = u_k(N) + \sum_{i=N}^{n-1} r_k(i) + \sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{kl}(i)\exp(u_l(i)). \tag{7}$$

Thus

$$\begin{aligned} c_k \le u_k(n) &\le u_k(N) + \sum_{i=N}^{n-1} r_k(i) + \sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{kl}(i)\exp(\alpha_l) \\ &\le u_k(N) + \sum_{i=N}^{n-1} r_k(i) + M\sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{kl}(i) < +\infty, \end{aligned}$$

where $M = \max_{1\le l\le m}\{\exp(\alpha_l)\}$.

Conversely, suppose that $B_k = \sum_{i=0}^{\infty}\sum_{l=1}^{m} b_{kl}(i) < \infty$ for $k = 1,\dots,m$. Note that (4) can be written as

$$u_k(n) = u_k(0) + \sum_{i=0}^{n-1} r_k(i) + \sum_{i=0}^{n-1}\sum_{l=1}^{m} b_{kl}(i)\exp(u_l(i)),\ \ k = 1,\dots,m. \tag{8}$$

Next, we can choose an integer N large enough so that

$$\sum_{i=N}^{\infty}\sum_{l=1}^{m} b_{kl}(i) \le \frac{d_k}{4\exp(d)},$$

where $d = \max_{1\le k\le m}\{d_k\}$, and $d_1,\dots,d_k$ are positive constants which satisfies

$$\sum_{i=N}^{\infty} r_k(i) \le \frac{d_k}{4},\ \ k = 1,\dots,m.$$

Let χ be the Banach space of all bounded real-valued sequences $\{u(n)\}_{n=N}^{\infty} = \{(u_1(n), \ldots, u_m(n))\}_{n=N}^{\infty}$ with the norm

$$\|(u_1(n), u_2(n), \cdots, u_m(n))\| = \max\left\{\sup_{n\geq N}|u_1(n)|, \sup_{n\geq N}|u_2(n), \cdots, \sup_{n\geq N}|u_m(n)|\right\},$$

and with the usual pointwise ordering $\leq$.

Define a subset Ψ of χ by

$$\Psi = \{\{u(n)\} \in \chi : \frac{d_k}{2} \leq u_k(n) \leq d_k, k = 1, \ldots, m, n \geq N\}.$$

For any subset B of Ψ, it is easy to see that $\inf B \in \Psi$ and $\sup B \in \Psi$. Define the operator $\mathbf{S} : \Psi \to \chi$ by

$$\mathbf{S}\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}(n) = \begin{pmatrix} \frac{d_1}{2} \\ \vdots \\ \frac{d_m}{2} \end{pmatrix} + \begin{pmatrix} \sum_{i=N}^{n-1} r_1(i) + \sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{1l}(i)\exp(u_l(i)) \\ \vdots \\ \sum_{i=N}^{n-1} r_m(i) + \sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{ml}(i)\exp(u_l(i)) \end{pmatrix}$$

for $n \geq N$. We claim that $\mathbf{S}$ maps Ψ into Ψ. To see this we let $\{u(n)\} \in \Psi$, then

$$\begin{aligned}
\frac{d_k}{2} \leq (\mathbf{S}u_k)(n) &= \frac{d_k}{2} + \sum_{i=N}^{n-1} r_k(i) + \sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{kl}(i)\exp(u_l(i)) \\
&\leq \frac{d_k}{2} + \frac{d_k}{4} + \sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{kl}(i)\exp(d_l) \\
&\leq \frac{d_k}{2} + \frac{d_k}{4} + \exp(d)\sum_{i=N}^{n-1}\sum_{l=1}^{m} b_{kl}(i) \leq d_k
\end{aligned}$$

for $n > N$. Since $\mathbf{S}$ is increasing, the mapping $\mathbf{S}$ satisfies the hypothesis of Knaster's fixed point theorem and hence we conclude that there exists $\{u(n)\} \in \Psi$ such that $\{u(n)\} = \mathbf{S}(\{u(n)\})$, that is,

$$u_k(n) = \frac{d_k}{2} + \sum_{i=0}^{n-1} r_k(i) + \sum_{i=0}^{n-1}\sum_{l=1}^{m} b_{kl}(i)\exp(u_l(i)),\ k \in 1, \ldots, m,$$

from which we obtain

$$\lim_{n\to\infty} u_k(n) = d_k^0,\ k = 1, \ldots, m,$$

where d_k^0 are positive constants. This completes the proof.

We now turn to the existence of solutions with only one convergent component sequence.

Theorem 3. *Suppose* (2) *holds. Suppose* $B_{j_0} = \sum_{i=0}^{\infty} \sum_{l=1}^{m} b_{j_0 l}(i) < +\infty$ *and* $B_k = \sum_{i=0}^{\infty} \sum_{l=1}^{m} b_{kl}(i) = +\infty$ *for* $k \in \{1, \ldots, m\} \backslash \{j_0\}$. *If there exists a positive solution* $\{u(n)\}$ *of* (4) *such that*

$$\lim_{n \to \infty} u_{j_0}(n) = \alpha_{j_0} < +\infty, \tag{9}$$

and

$$\lim_{n \to \infty} u_k(n) = \alpha_k = +\infty, \ k \in \{1, .., m\} \backslash \{j_0\}, \tag{10}$$

then

$$\sum_{i=0}^{\infty} \sum_{l=1}^{m} b_{j_0 l}(i) \exp[\sum_{h=0}^{i-1} b_{l j_0}(h) \exp(c)] < +\infty, \tag{11}$$

for some positive constant c. *Conversely, if* (11) *holds, then* (4) *has an eventually positive solution* $\{u(n)\}$ *that satisfies* (9) *and* (10).

Proof. Let $\{u(n)\}$ be a positive solution of (4). Then each component sequence $\{u_k(n)\}$ is increasing, and then there exists a positive constant β such that $u_{j_0}(0) \leq u_{j_0}(n) \leq \beta$ for $n \geq 0$. From (4) we have

$$\begin{aligned} u_k(n) &= u_k(0) + \sum_{i=0}^{n-1} r_k(i) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{kl}(i) \exp(u_l(i)) \\ &\geq \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{kl}(i) \exp(u_l(0)), \end{aligned}$$

for $k \neq j_0$. On the other hand, we have

$$\begin{aligned} \beta \geq u_{j_0}(n) &= u_{j_0}(0) + \sum_{i=0}^{n-1} r_{j_0}(i) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{j_0 l}(i) \exp(u_l(i)) \\ &\geq \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{j_0 l}(i) \exp(u_l(i)) \geq \sum_{i=0}^{n-1} \sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{ \sum_{h=0}^{i-1} b_{l j_0}(h) \exp(u_{j_0}(0)) \right\}. \end{aligned}$$

By taking the limit at infinity in the above inequality, we obtain (11).

Conversely, suppose that (11) holds. We can choose an integer N large enough so that

$$\sum_{i=n}^{\infty} \sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{ \sum_{h=N}^{i-1} b_{l j_0}(h) \right\} < \frac{D}{4 \exp(D)}, \quad n > N, \tag{12}$$

and

$$\sum_{i=n}^{\infty} r_{j_0}(i) \leq \frac{D}{4}, \quad n > N.$$

Let χ be the set of all bounded real-valued scalar sequences of the form $w = \{w(n)\}_{i=N}^{\infty}$ with the norm

$$\|w\| = \sup_{n \geq N} |w(n)|\},$$

Then χ is a Banach space. Define a subset Ψ of χ by

$$\Psi = \{\{w(n)\} \in \chi : \frac{D}{2} \le w(n) \le D, n \ge N\}.$$

Then Ψ is a bounded, convex and closed subset of χ . Define the operator $\mathbf{E} : \Psi \to \chi$ by

$$(\mathbf{Ew})(n) = D - \sum_{i=n}^{\infty} r_{j_0}(i) - \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=0}^{i-1} b_{l j_0}(h) \exp(w(h))\right\}, \quad n \ge N. \quad (13)$$

First, $\mathbf{E}$ maps Ψ into Ψ since

$$\begin{aligned} D \ge (\mathbf{E}w)(n) &= D - \sum_{i=n}^{\infty} r_{j_0}(i) - \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp[\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(w(h))] \\ &\ge D - \frac{D}{4} - \exp(D) \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h)\right\} \ge \frac{D}{2}. \end{aligned}$$

Next, we show that $\mathbf{E}$ is continuous. Let $\{w^s\}$ be a sequence in Ψ such that

$$\lim_{s\to\infty} \|w^s - w\| = 0.$$

Since Ψ is closed, $w \in \Psi$. Then by (13), we have

$$\begin{aligned} &|(\mathbf{E}w^s)(n) - (\mathbf{E}w)(n)| \\ &= \left| \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(w^s(h))\right\} \right. \\ &\quad \left. - \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(w(h))\right\} \right| \\ &\le \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \left| \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(w^s(h))\right\} - \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(w(h))\right\} \right|. \end{aligned}$$

By the continuity of the exponential function and the Lebesgue dominated convergence theorem, it follows that

$$\lim_{s\to\infty} \sup_{n\ge N} |(\mathbf{E}w^s)(n) - (\mathbf{E}w)(n)| = 0.$$

This shows that $\mathbf{E}$ is continuous.

Finally, we show that $\mathbf{E}\Psi$ is precompact. Let $w \in \Psi$ and $k > n \ge N$, then in view of (12),

$$\begin{aligned} |(\mathbf{E}w)(k) - (\mathbf{E}w)(n)| &\le \sum_{i=n}^{k-1}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(w(h))\right\} \\ &\le \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=N}^{i-1} b_{l j_0}(h) \exp(D)\right\}. \end{aligned}$$

This means that $\mathbf{E}\Psi$ is precompact.

Now, by Schauder's fixed point theorem, we conclude that there exists $w' \in \Psi$ such that $w'(n) = (\mathbf{E}w')(n)$, that is,

$$w'(n) = D - \sum_{i=n}^{\infty} r_{j_0}(i) - \sum_{i=n}^{\infty}\sum_{l=1}^{m} b_{j_0 l}(i) \exp\left\{\sum_{h=N}^{i-1} b_{lj_0}(h) \exp(w'(h))\right\}, \ n > N,$$

from which we obtain

$$\lim_{n\to\infty} w'(n) = d^0,$$

where d^0 is a positive constant. On the other hand, set $u_{j_0}(n) = w'(n)$ and

$$u_k(n) = \sum_{i=0}^{n-1} r_k(i) + \sum_{i=0}^{n-1}\sum_{l=1}^{m} b_{kl}(i) \exp(u_l(i)), \quad k \neq j_0.$$

Then

$$\triangle u_k(n) = r_k(n) + \sum_{l=1}^{m} b_{kl}(n) \exp(u_l(n)), \ k = 1, \ldots, m,$$

and

$$u_k(n) \geq \sum_{i=0}^{n-1}\sum_{l=1}^{m} b_{kl}(i) \exp(u_l(0)) \to +\infty$$

for $k \neq j_0$. The proof is complete.

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