SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract: We introduce some subclasses of close-to-convex functions and obtain sharp results for coefficients, distortion theorems and argument theorems from which results of several authors follows as special cases.

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1. Introduction and Definitions

Principle of Subordination([9], [13]). Let $f(z)$ and $F(z)$be two functions analytic in the open unit disc $E = \{z;\ |z| < 1\}$. Then $f(z)$is subordinate to $F(z)$ in E if there exists a function $w(z)$ analytic in E and satisfying the condition $w(z) = 0$, $|w(z)| < 1$ such that $f(z) = F\big(w(z)\big)$. If $F(z)$ is univalent in E, the above definition is equivalent to $f(0) = F(0)$ and $f(E) \subset F(E)$.

Bounded Functions. By $\mathcal{U}$, we denote the class of analytic functions of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in E, \tag{1.1}$$

which satisfy the conditions $w(z) = 0$ and $|w(z)| < 1$.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

which are analytic in the unit disc $E = \{z;\ |z| < 1\}$. The subclass of univalent functions in $\mathcal{A}$ is denoted by S.

S^* and C represent the classes of functions in $\mathcal{A}$ which satisfy, respectively, the conditions

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \tag{1.3}$$

$$\text{Re}\left\{\frac{\big(zf'(z)\big)'}{f'(z)}\right\} > 0. \tag{1.4}$$

A function $f(z)$ in $\mathcal{A}$ is said to be close-to-convex if there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.5}$$

in S^* such that

$$\text{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0. \tag{1.6}$$

The class of functions $f(z)$ in $\mathcal{A}$ with the condition (1.6) is denoted by K and called the class of close-to- convex functions. The class K was introduced by Kaplan [7] it was shown by him that all close-to-convex functions are univalent.

If $g \in C$, the class of functions in $\mathcal{A}$ subject to the condition (1.6) may be denoted by K_1 which is the subclass of K.

$S^*(A, B)$ and $C(A, B)$ are the classes of functions in $\mathcal{A}$ which satisfy, respectively, the conditions

(1.7) $$\frac{zg'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in S^*, \quad -1 \le B < A \le 1,$$

(1.8) $$\frac{\left(zg'(z)\right)'}{g'(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in C, \quad -1 \le B < A \le 1.$$

In particular, $S^*(1, -1) \equiv S^*$ and $C(1, -1) \equiv C$.

The class $S^*(A, B)$ was introduced and study by Janowski [6] and also by Goel and the first author [4]. It is obvious that $g \in C(A, B)$ implies that $zg'(z) \in S^*(A, B)$.

$K(C, D)$ represent the class of functions $f(z)$ in $\mathcal{A}$ for which

(1.9) $$\frac{zf'(z)}{g(z)} \prec \frac{1+Cz}{1+Dz}, \quad g \in S^*, \quad -1 \le D < C \le 1.$$

If $g \in C$, the corresponding class may be denoted by $K_1(C, D)$.

The class $K^*(A, B)$ consists of functions $f(z)$ in $\mathcal{A}$ such that

(1.10) $$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad g \in S^*(A, B), \quad -1 \le B < A \le 1.$$

If $g \in C(A, B)$, the corresponding class may be denoted by $K_1^*(A, B)$.

For $-1 \le D \le B < A \le C \le 1$, let $K(A, B; C, D)$ be the subclass of K satisfying

(1.11) $$\frac{zf'(z)}{g(z)} \prec \frac{1+Cz}{1+Dz}, \quad g \in S^*(A, B).$$

If $g \in C(A, B)$, the corresponding class may be denoted by $K_1(A, B; C, D)$.

Throughout the paper, we take $-1 \le D \le B < A \le C \le 1$, $w(z) \in \mathcal{U}$ and $z \in E$. From the above definitions, we have the following observations:

(i) $K(1, -1; C, D) \equiv K(C, D)$ and $K_1(1, -1; C, D) \equiv K_1(C, D)$;

(ii) $K(A, B; 1, -1) \equiv K^*(A, B)$ and $K_1(A, B; 1, -1) \equiv K_1^*(A, B)$;

(iii) $K(1, -1; 1, -1) \equiv K$ and $K_1(1, -1; 1, -1) \equiv K_1$.

2. Preliminary Lemmas

Lemma 2.1 [3]. Let $P(z) = \frac{1+Cw(z)}{1+Dw(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n$, then

$$|p_n| \le (C - D).$$

Result is sharp for the functions $P_n(z) = \frac{1+C\delta z^n}{1+D\delta z^n}$, $|\delta| = 1$ and $n \ge 1$.

Lemma 2.2 [4]. Let $g \in S^*(A, B)$, then, for $A - (n-1)B \ge (n-2)$, $(n \ge 3)$,

$$|b_n| \le \frac{1}{(n-1)!}\prod_{k=2}^{n}(A - (k-1)B).$$

Equality holds for the function $g_0(z)$ defined by

$$g_0(z) = z(1 + B\delta z)^{(A-B)/B}, \quad |\delta| = 1.$$

Since $g(z) \in C(A, B)$ implies that $zg'(z) \in S^*(A, B)$, we have the following

Lemma 2.3. Let $g \in C(A, B)$, then, for $A - (n-1)B \ge (n-2)$, $(n \ge 3)$,

$$|b_n| \le \frac{1}{n!}\prod_{k=2}^{n}(A - (k-1)B).$$

Result is sharp for the function $g_1(z)$ defined by

$$g_1'(z) = (1 + B\delta z)^{(A-B)/B}, \quad |\delta| = 1.$$

Lemma 2.4 [5]. Let $g \in S^*(A, B)$, then, for $|s| \le 1$, $|t| \le 1$, $(s \ne t)$

$$\frac{tg(sz)}{sg(tz)} \prec \begin{cases} \left(\frac{1+Bsz}{1+Btz}\right)^{(A-B)/B}, & B \neq 0; \\ \exp A(s-t)z, & B = 0. \end{cases}$$

Lemma 2.5. If $g \in S^*(A,B)$, then, for $|z| = r < 1$,

(2.1) $\quad r(1-Br)^{(A-B)/B} \leq |g(z)| \leq r(1+Br)^{(A-B)/B}, B \neq 0;$

(2.2) $\quad r\exp(-Ar) \leq |g(z)| \leq r\exp(Ar), B = 0;$

(2.3) $\quad \left|\arg\frac{g(z)}{z}\right| \leq \frac{(A-B)}{B}\sin^{-1}(Br), B \neq 0;$

(2.4) $\quad \left|\arg\frac{g(z)}{z}\right| \leq Ar, B = 0.$

Equality sign in these bounds is attained by the function $g_0(z)$ defined by

$$g_0(z) = \begin{cases} z(1+B\delta z)^{(A-B)/B}, & B \neq 0; \\ z\exp(A\delta z), & B = 0, |\delta| = 1. \end{cases}$$

Proof. Letting $s \to 1$ and $t \to 0$ in the Lemma 2.4, we obtain

(2.5) $\quad \frac{g(z)}{z} \prec (1+Bz)^{(A-B)/B}, \ B \neq 0;$

(2.6) $\quad \frac{g(z)}{z} \prec \exp(Az), \quad B = 0.$

(2.5) implies that

(2.7) $\quad \frac{g(z)}{z} = (1+Bw(z))^{(A-B)/B}, \ B \neq 0.$

Case (i) $B > 0$.

$$\begin{aligned} \left|\left(1+Bw(z)\right)^{(A-B)/B}\right| &= \left|\exp\left\{\frac{(A-B)}{B}\log\left(1+Bw(z)\right)\right\}\right| \\ &= \exp\left\{\frac{(A-B)}{B}\log|1+Bw(z)|\right\} \\ &\leq |1+Bw(z)|^{(A-B)/B} \leq (1+Br)^{(A-B)/B}. \end{aligned}$$

Case (ii) $B < 0$. Let $B = -B'$, $B' > 0$. Then

$$\begin{aligned} \left|\left(1+Bw(z)\right)^{(A-B)/B}\right| &= \left|\left\{\left(1-B'w(z)\right)^{-1}\right\}^{(A-B)/B'}\right| \\ &= \left|\left(1-B'w(z)\right)^{-1}\right|^{(A-B)/B'} \\ &\leq \left(\frac{1}{1-B'r}\right)^{(A-B)/B'} = (1+Br)^{(A-B)/B}. \end{aligned}$$

Combining the cases (i) and (ii), (2.1) follows from (2.7). Similarly, we get (2.2) from (2.6). Again from (2.5), we obtain (2.3) as follows

$$\left|\arg\frac{g(z)}{z}\right| \leq \frac{(A-B)}{B}\left|\arg\left(1+Bw(z)\right)\right| \leq \frac{(A-B)}{B}\sin^{-1}(Br).$$

Similarly (2.4) directly follows from (2.6).

On the same lines we can prove the following

Lemma 2.6. If $g \in C(A,B)$, then, for $|z| = r < 1$,

$$\frac{1}{A}\left\{1-(1-Br)^{A/B}\right\} \leq |g(z)| \leq \frac{1}{A}\left\{(1+Br)^{A/B}-1\right\}, B \neq 0;$$

$$\frac{1}{A}\exp(-Ar) \leq |g(z)| \leq \frac{1}{A}\exp(Ar), B = 0;$$

$$\left|\arg\frac{g(z)}{z}\right| \leq \frac{A}{B}\sin^{-1}(Br), B \neq 0;$$

$$\left|\arg\frac{g(z)}{z}\right| \le \mathrm{Ar}, \mathrm{B}=0.$$

Lemma 2.7 [2]. Let f and g are analytic functions and h be convex univalent function in E such that $f \prec h$ and $g \prec h$. Then $(1-\lambda)f+\lambda g \prec h,\ (0 \le \lambda \le 1)$.

Lemma 2.8 Let $g \in \mathrm{C(A,B)}$, then, for $|z| = r < 1$,

$$\text{(2.8)} \qquad \mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{\mathrm{Ar}(1-\mathrm{Br})^{(\mathrm{A}-\mathrm{B})/\mathrm{B}}}{1-(1-\mathrm{Br})^{\mathrm{A}/\mathrm{B}}}.$$

Proof. Since $g \in \mathrm{C(A,B)}$ implies that $zg'(z) \in \mathrm{S}^*(\mathrm{A,B})$, we have

$$zg'(z) = z(1+\mathrm{B}z)^{(\mathrm{A}-\mathrm{B})/\mathrm{B}}$$

which yield

$$g(z) = \frac{1}{\mathrm{A}}\left\{(1+\mathrm{B}z)^{\mathrm{A}/\mathrm{B}}-1\right\}.$$

Therefore

$$\frac{zg'(z)}{g(z)} = \frac{\mathrm{A}z(1+\mathrm{B}z)^{(\mathrm{A}-\mathrm{B})/\mathrm{B}}}{(1+\mathrm{B}z)^{\mathrm{A}/\mathrm{B}}-1}$$

which implies (2.8).

3. Coefficient Estimates

Theorem 3.1. Let $f \in \mathrm{K(A,B;C,D)}$. Then, for $\mathrm{A}-(\mathrm{n}-1)\mathrm{B} \ge (\mathrm{n}-2), (\mathrm{n} \ge 3)$,

$$\text{(3.1)} \quad |a_n| \le \frac{1}{n!}\prod_{k=2}^{n}\{\mathrm{A}-(k-1)\mathrm{B}\} + \frac{(\mathrm{C}-\mathrm{D})}{n}\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\{\mathrm{A}-(j-1)\mathrm{B}\}\right).$$

Bound (3.1) is sharp.

Proof. By definition of $\mathrm{K(A,B;C,D)}$,

$$\frac{zf'(z)}{g(z)} = \frac{1+\mathrm{C}w(z)}{1+\mathrm{D}w(z)} = P(z).$$

Expanding the series,

$$\text{(3.2)} \quad (z+2a_2z^2+\cdots+na_nz^n+\cdots)$$
$$= (z+b_2z^2+\cdots+b_{n-1}z^{n-1}+b_nz^n+\cdots)(1+p_1z+p_2z^2+\cdots+p_{n-1}z^{n-1}+\cdots)$$

Equating the coefficients of z^n in (3.2),

$$na_n = b_n + p_1b_{n-1} + p_2b_{n-2} + \cdots + p_{n-2}b_2 + p_{n-1}$$

Applying triangular inequality and Lemma 2.1 , we get

$$\text{(3.3)} \quad n|a_n| \le |b_n| + (\mathrm{C}-\mathrm{D})\left(1+\sum_{k=2}^{n-1}|b_k|\right).$$

Using Lemma (2.2) in (3.3), we obtain

$$n|a_n| \le \frac{1}{(n-1)!}\prod_{k=2}^{n}\{\mathrm{A}-(k-1)\mathrm{B}\} + (\mathrm{C}-\mathrm{D})\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\{\mathrm{A}-(j-1)\mathrm{B}\}\right)$$

which yields (3.1). The bound (3.1) is sharp for the function $f_0(z)$ defined by

$$f_0(z) = \left(\frac{1+\mathrm{C}\delta_1 z}{1+\mathrm{D}\delta_1 z}\right)(1+\mathrm{B}\delta_2 z)^{(\mathrm{A}-\mathrm{B})/\mathrm{B}}, \qquad |\delta_1| = |\delta_2| = 1.$$

Similarly we can prove

Theorem 3.2. Let $f \in \mathrm{K}_1(\mathrm{A,B;C,D})$. Then, for $\mathrm{A}-(\mathrm{n}-1)\mathrm{B} \ge (\mathrm{n}-2), (\mathrm{n} \ge 3)$,

$$\text{(3.4)} \quad |a_n| \le \frac{1}{n}\left[\frac{1}{n!}\prod_{k=2}^{n}(\mathrm{A}-(k-1)\mathrm{B}) + \frac{(\mathrm{C}-\mathrm{D})}{n}\left(1+\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}\{\mathrm{A}-(j-1)\mathrm{B}\}\right)\right].$$

The bound (3.4) is sharp for the function $f_1(z)$ given by

$$f_1'(z) = \frac{1}{Az}\left(\frac{1 + C\delta_1 z}{1 + D\delta_1 z}\right)\{(1 + B\delta_2 z)^{A/B} - 1\}, \qquad |\delta_1| = |\delta_2| = 1.$$

Remark 3.1 (i) If $f \in K(1, -1; C, D) \equiv K(C, D)$, $|a_n| \leq 1 + \frac{(n-1)(C-D)}{n}$ which is the result due to first author [10].

Remark 3.1 (ii) If $f \in K_1(1, -1; C, D) \equiv K_1(C, D)$, $|a_n| \leq \frac{1}{n}\{1 + (n-1)(C-D)\}$, a result due to first author and G. Singh [11].

Remark 3.2 (i) If $f \in K(A, B; 1, -1) \equiv K^*(A, B)$, for $A - (n-1)B \geq (n-2)$, $(n \geq 3)$,

$$|a_n| \leq \frac{1}{n!}\prod_{k=2}^{n}\{A - (k-1)B\} + \frac{2}{n}\left(1 + \sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\{A - (j-1)B\}\right).$$

This result was proved by Goel and the first author [5].

Remark 3.2 (ii) If $f \in K_1(A, B; 1, -1) \equiv K_1^*(A, B)$, for $A - (n-1)B \geq (n-2)$, $(n \geq 3)$,

$$|a_n| \leq \frac{1}{n(n!)}\prod_{k=2}^{n}\{A - (k-1)B\} + \frac{2}{n^2}\left(1 + \sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}\{A - (j-1)B\}\right).$$

Remark 3.3 (i) If $f \in K(1, -1; 1, -1) \equiv K$, then $|a_n| \leq n$. The result due to Reade [13].

Remark 3.3 (ii) If $f \in K_1(1, -1; 1, -1) \equiv K_1$, then $|a_n| \leq 2 - \frac{1}{n}$. This result was obtained by Silverma and Telage [15].

4. Distortion Theorems

Theorem 4.1 Let $f \in K(A, B; C, D)$, then

(4.1) $$\left(\frac{1 - Cr}{1 - Dr}\right)(1 - Br)^{(A-B)/B} \leq |f'(z)| \leq \left(\frac{1 + Cr}{1 + Dr}\right)(1 + Br)^{(A-B)/B}, B \neq 0;$$

(4.2) $$\left(\frac{1 - Cr}{1 - Dr}\right)\exp(-Ar) \leq |f'(z)| \leq \left(\frac{1 + Cr}{1 + Dr}\right)\exp(Ar), B = 0;$$

(4.3) $$\int_0^r \left(\frac{1 - Cu}{1 - Du}\right)(1 - Bu)^{(A-B)/B}du \leq |f(z)| \leq \int_0^r \left(\frac{1 + Cu}{1 + Du}\right)(1 + Bu)^{(A-B)/B}du, B \neq 0;$$

(4.4) $$\int_0^r \left(\frac{1 - Cu}{1 - Du}\right)\exp(-Au)du \leq |f(z)| \leq \int_0^r \left(\frac{1 + Cu}{1 + Du}\right)\exp(Au)du, B = 0.$$

All these bounds are sharp.

Proof. Since $f \in K(A, B; C, D)$, it follows that

$$\frac{zf'(z)}{g(z)} = \frac{1 + Cw(z)}{1 + Dw(z)}$$

which maps $|w(z)| \leq r$ onto the circle

$$\left|\frac{zf'(z)}{g(z)} - \frac{1 - CDr^2}{1 - D^2r^2}\right| \leq \frac{(C - D)r}{(1 - D^2r^2)}.$$

This yields

$$\frac{(1 - Cr)}{(1 - Dr)} \leq \left|\frac{zf'(z)}{g(z)}\right| \leq \frac{(1 + Cr)}{(1 + Dr)}$$

which further implies that

(4.5) $$\frac{(1 - Cr)}{(1 - Dr)}|g(z)| \leq |zf'(z)| \leq \frac{(1 + Cr)}{(1 + Dr)}|g(z)|.$$

Using (2.1) and (2.2) along with (4.5), we obtain (4.1) and (4.2).
Now

$$|f(z)| = \left|\int_0^z f'(z)dz\right| \le \int_0^r |f'(z)|dr \le \begin{cases} \int_0^r \left(\frac{1+Cr}{1+Dr}\right)(1+Br)^{(A-B)/B}dr, B \ne 0; \\ \int_0^r \left(\frac{1+Cr}{1+Dr}\right)\exp(Ar)dr, \quad B = 0. \end{cases}$$

Let z_0, $|z_0| = 1$, be so chosen that $|f(z_0)| \le |f(z)|$ for all z, $|z| = r$. If $L(z_0)$ is the pre-image of the segment $\overline{[0, f(z_0)]}$ in E, then

$$|f(z_0)| = \int_{L(z_0)} |f'(z)|dr \ge \begin{cases} \int_0^r \left(\frac{1-Cr}{1-Dr}\right)(1-Br)^{(A-B)/B}dr, B \ne 0; \\ \int_0^r \left(\frac{1-Cr}{1-Dr}\right)\exp(-Ar)dr, \quad B = 0. \end{cases}$$

Equality signs in (4.1), (4.2), (4.3) and (4.4) are attained by the function $f_2(z)$ defined by

(4.6) $$(f_2(z))' = \begin{cases} \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)(1+B\delta_2 z)^{(A-B)/B}, \quad B \ne 0; \\ \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)\exp(A\delta_2 z), B = 0, |\delta_1| = |\delta_2| = 1. \end{cases}$$

Similarly, by using Lemma 2.6, we can prove

Theorem 4.2 Let $f \in K_1(A, B; C, D)$, then

$$\frac{1}{Ar}\left(\frac{1-Cr}{1-Dr}\right)\{1-(1-Br)^{A/B}\} \le |f'(z)| \le \frac{1}{Ar}\left(\frac{1+Cr}{1+Dr}\right)\{(1+Br)^{A/B}-1\}, B \ne 0;$$

$$\frac{1}{Ar}\left(\frac{1-Cr}{1-Dr}\right)\exp(-Ar) \le |f'(z)| \le \frac{1}{Ar}\left(\frac{1+Cr}{1+Dr}\right)\exp(Ar), \quad B = 0;$$

$$\frac{1}{A}\int_0^r \frac{1}{u}\left(\frac{1-Cu}{1-Du}\right)\{1-(1-Bu)^{A/B}\}du \le |f(z)| \le \frac{1}{A}\int_0^r \frac{1}{u}\left(\frac{1+Cu}{1+Du}\right)\{(1+Bu)^{A/B}-1\}du, B \ne 0;$$

$$\frac{1}{A}\int_0^r \frac{1}{u}\left(\frac{1-Cu}{1-Du}\right)\exp(-Au)du \le |f(z)| \le \frac{1}{A}\int_0^r \frac{1}{u}\left(\frac{1+Cu}{1+Du}\right)\exp(Au)du, B = 0.$$

All these bounds are sharp and extremal function is given by $f_3(z)$ defined by

(4.7) $$(f_3(z))' = \begin{cases} \frac{1}{Az}\left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)\{(1+B\delta_2 z)^{A/B}-1\}, \quad B \ne 0; \\ \frac{1}{Az}\left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)\exp(A\delta_2 z), \quad B = 0, |\delta_1| = |\delta_2| = 1. \end{cases}$$

5. Argument Theorems

Theorem 5.1 Let $f \in K(A, B; C, D)$, then

(5.1) $$|\arg f'(z)| \le \frac{(A-B)}{B}\sin^{-1}(Br) + \sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}, B \ne 0;$$

(5.2) $$|\arg f'(z)| \le Ar + \sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}, \quad B = 0.$$

The results are sharp.

Proof. From (1.11), we have $\frac{zf^{'}(z)}{g(z)}=\frac{1+Cw(z)}{1+Dw(z)}$.

Since the transformation $\frac{zf^{'}(z)}{g(z)}=\frac{1+Cw(z)}{1+Dw(z)}$ maps $|w(z)|\leq r$ onto the circle

$$\left|\frac{zf^{'}(z)}{g(z)}-\frac{1-CDr^2}{1-D^2r^2}\right|\leq\frac{(C-D)r}{(1-D^2r^2)},$$

therefore

$$\left|\arg\frac{zf^{'}(z)}{g(z)}\right|\leq\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}.$$

This implies that

$$|\arg f^{'}(z)|\leq\left|\arg\frac{g(z)}{z}\right|+\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}. \tag{5.3}$$

(5.3) together with (2.3) and (2.4) yield (5.1) and (5.2) respectively. Equality sign in (5.1) and (5.2) holds for the function $f_2(z)$ defined by (4.6) in which

$$\delta_1=\frac{r}{z}\left[\frac{-(C+D)r+i\{(1-C^2r^2)(1-D^2r^2)\}^{1/2}}{(1+CDr^2)}\right] \tag{5.4}$$

and

$$\delta_2=\frac{r}{z}\{-Br+i(1-B^2r^2)^{1/2}\}. \tag{5.5}$$

Similarly, by using Lemma 2.6, we have

Theorem 5.2 Let $f\in K_1(A,B;C,D)$, then

$$|\arg f^{'}(z)|\leq\frac{A}{B}\sin^{-1}(Br)+\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}, B\neq 0;$$

$$|\arg f^{'}(z)|\leq Ar+\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\},\qquad B=0.$$

The results are sharp for the function $f_3(z)$ defined in (4.7) where δ_1 and δ_1 are given by (5.4) and (5.5), respectively.

Remark 5.1 Taking $A=1$ and $B=-1$ in the Theorem 4.1, we get the result proved by the first author [10].

Remark 5.2 On taking $C=1$ and $D=-1$ in the Theorems 4.1 and 5.1, we get the results due to Goel and the first author [5].

Remark 5.3 Letting $A=C=1$ and $B=D=-1$ in the Theorems 4.1 and 5.1, we obtain the results proved by Ogawa [12] and Krzyz [8] for the class K.

Remark 5.4 For $C=1$ and $D=-1$ in Theorems 4.2 and 5.2, we get the results established by Gawad and Thomas [1].

6. Convex Set of Functions

Theorem 6.1 If f and $h\in K(A,B;C,D)$, then

$$(1-\lambda)f+\lambda h\in K(A,B;C,D),\quad(0\leq\lambda\leq 1).$$

Proof. Since $\frac{1+Cz}{1+Dz}$ is convex univalent in E, the theorem follows by Lemma 2.7 definition of $K(A,B;C,D)$.

7. Radius of Convexity for $K^*(A,B)$ and $K_1^*(A,B)$

Theorem 7.1 If $f\in K^*(A,B)$, then $f(z)$ is convex in $|z|\leq r_0$ where r_0 is the smallest positive root of the equation

(7.1) $$\mathrm{A}r^3-(1-2\mathrm{B})r^2-(2+\mathrm{A})r+1=0.$$
The radius r_0 is sharp.
Proof. Since $f\in K^*(A,B)$, for $g\in S^*(A,B)$ we have
(7.2) $$zf'(z)=g(z)P(z).$$
Differentiating (7.1) logarithmically, we get
$$1+\frac{zf'(z)}{f(z)}=\frac{zg'(z)}{g(z)}+\frac{zP'(z)}{P(z)}$$
which implies that
(7.3) $$Re\left\{1+\frac{zf'(z)}{f(z)}\right\}\geq Re\left\{\frac{zg'(z)}{g(z)}\right\}-\left|\frac{zP'(z)}{P(z)}\right|.$$
Since $g\in S^*(A,B)$, it follows that $\frac{zg'(z)}{g(z)}=\frac{1+Aw(z)}{1+Bw(z)}$ from which it is easily verify that for $|w(z)|\leq r$,
(7.4) $$\frac{1-\mathrm{A}r}{1-\mathrm{B}r}\leq Re\left\{\frac{zg'(z)}{g(z)}\right\}\leq\frac{1+\mathrm{A}r}{1+\mathrm{B}r}.$$
Also it is known that
(7.5) $$\left|\frac{zP'(z)}{P(z)}\right|\leq\frac{2r}{1-r^2}.$$
From (7.3), (7.4) and (7.5), it is deduce that $Re\left\{1+\frac{zf'(z)}{f(z)}\right\}>0$, provided $|z|\leq r_0$ where r_0 is the smallest positive root of the equation (7.1). Sharp result is obtained for the function $f_0(z)$ defined by
$$f_0'(z)=\left(\frac{1+\delta_1 z}{1-\delta_1 z}\right)\{1+\mathrm{B}\delta_2 z\}^{(\mathrm{A}-\mathrm{B})/\mathrm{B}},\ \ |\delta_1|=|\delta_2|=1.$$
Remark 7.1 Taking A=1 and B= -1 in the Theorem, we get $r_0=2-\sqrt{3}$ which is the radius of convexity for the class of convex functions.
Theorem 7.2 If $f\in K_1^*(A,B)$, then $f(z)$ is convex in $|z|\leq r_1$ where r_1 is the smallest positive root of the equation
(7.6) $$2\left\{1-(1-\mathrm{B}r)^{\mathrm{A}/\mathrm{B}}\right\}-\mathrm{A}(1-r^2)(1-\mathrm{B}r)^{(\mathrm{A}-\mathrm{B})/\mathrm{B}}=0.$$
The radius r_1 is sharp.
Proof. Since $f\in K_1^*(A,B)$, for $g\in C(A,B)$ we have
(7.7) $$zf'(z)=g(z)P(z).$$
Differentiating (7.7) logarithmically, we get
$$1+\frac{zf'(z)}{f(z)}=\frac{zg'(z)}{g(z)}+\frac{zP'(z)}{P(z)}$$
which implies that
(7.8) $$Re\left\{1+\frac{zf'(z)}{f(z)}\right\}\geq Re\left\{\frac{zg'(z)}{g(z)}\right\}-\left|\frac{zP'(z)}{P(z)}\right|.$$
Using Lemma 2.8 and (7.5) in (7.8), we obtain
$$Re\left\{1+\frac{zf'(z)}{f(z)}\right\}\geq\frac{\mathrm{A}r(1-\mathrm{B}r)^{(\mathrm{A}-\mathrm{B})/\mathrm{B}}}{1-(1-\mathrm{B}r)^{\mathrm{A}/\mathrm{B}}}-\frac{2r}{1-r^2}.$$
This impies that $Re\left\{1+\frac{zf'(z)}{f(z)}\right\}>0$, provided $|z|\leq r_1$ where r_1 is the smallest positive root of the equation (7.6). Sharp result is obtained for the function $f_1(z)$ defined by
$$f_1'(z)=\frac{1}{\mathrm{A}z}\left(\frac{1+\delta_1 z}{1-\delta_1 z}\right)\left\{(1+\mathrm{B}\delta_2 z)^{\mathrm{A}/\mathrm{B}}-1\right\},\ \ |\delta_1|=|\delta_2|=1.$$

Remark 7.2 Taking A=1 and B= -1 in the Theorem, we get $r_1 = \frac{1}{3}$ which is the radius of convexity for the class K_1. This result was due to Gawad and Thomas [1].

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