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SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. We introduce some subclasses of close-to-convex functions and obtain sharp results for coefficients, distortion theorems and argument theorems from which results of several authors follows as special cases.

1. Introduction and Definitions

Principle of Subordination ([9], [13]). Let $f(z)$ and $F(z)$ be two functions analytic in the open unit disc $E=\{z;|z|<1\}$. Then $f(z)$ is subordinate to $F(z)$ in E if there exists a function $w(z)$ analytic in E and satisfying the conditions $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z))$. If $F(z)$ is univalent in E, the above definition is equivalent to $f(0)=F(0)$ and $f(E)\subset F(E)$.

Bounded Functions. By $\mathscr{U}$, we denote the class of analytic functions of the form

$$w(z)=\sum_{n=1}^{\infty}c_n z^n, \quad z\in E, \tag{1.1}$$

which satisfy the conditions $w(0)=0$ and $|w(z)|<1$.

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.2}$$

which are analytic in the unit disc $E=\{z;|z|<1\}$. The subclass of univalent functions in $\mathscr{A}$ is denoted by S.

S^* and C represent the classes of functions in $\mathscr{A}$ which satisfy, respectively, the conditions

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0, \tag{1.3}$$

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$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} > 0. \tag{1.4}$$

A function $f(z)$ in $\mathscr{A}$ is said to be close-to-convex if there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.5}$$

in S^* such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0. \tag{1.6}$$

The class of functions $f(z)$ in $\mathscr{A}$ with the condition (1.6) is denoted by K and called the class of close-to-convex functions. The class K was introduced by Kaplan [7] it was shown by him that all close-to-convex functions are univalent.

If $g \in C$, the class of functions in $\mathscr{A}$ subject to the condition (1.6) may be denoted by K_1 which is the subclass of K.

$S^*(A,B)$ and $C(A,B)$ are the classes of functions in $\mathscr{A}$ which satisfy, respectively, the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \prec \frac{1+Az}{1+Bz}, \quad g \in S^*, \quad -1 \le B < A \le 1, \tag{1.7}$$

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} \prec \frac{1+Az}{1+Bz}, \quad g \in C, \quad -1 \le B < A \le 1. \tag{1.8}$$

In particular, $S^*(1,-1) \equiv S^*$ and $C(1,-1) \equiv C$.

The class $S^*(A,B)$ was introduced and study by Janowski [6] and also by Goel and Mehrok [4]. It is obvious that $g \in C(A,B)$ implies that $zg'(z) \in S^*(A,B)$.

$K(C,D)$ represent the class of functions $f(z)$ in $\mathscr{A}$ for which

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \prec \frac{1+Cz}{1+Dz}, \quad g \in S^*, \quad -1 \le D < C \le 1. \tag{1.9}$$

If $g \in C$, the corresponding class may be denoted by $K_1(C,D)$.

The class $K^*(A,B)$ consists of functions $f(z)$ in $\mathscr{A}$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad g \in S^*(A,B), \quad -1 \le B < A \le 1. \tag{1.10}$$

If $g \in C(A,B)$, the corresponding class may be denoted by $K_1^*(A,B)$.

For $-1 \le D \le B < A \le C \le 1$, let $K(A,B;C,D)$ be the subclass of K satisfying

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} \prec \frac{1+Cz}{1+Dz}, \quad g \in S^*(A,B). \tag{1.11}$$

If $g \in C(A,B)$, the corresponding class may be denoted by $K_1(A,B;C,D)$.

Throughout the paper, we take $-1 \le D \le B < A \le C \le 1$, $w(z) \in \mathscr{U}$ and $z \in E$. From the above definitions, we have the following observations

(i) $K(1,-1;C,D) \equiv K(C,D)$ and $K_1(1,-1;C,D) \equiv K_1(C,D)$;

(ii) $K(A,B;1,-1) \equiv K^*(A,B)$ and $K_1(A,B;1,-1) \equiv K_1^*(A,B)$;

(iii) $K(1,-1;1,-1) \equiv K$ and $K_1(1,-1;1,-1) \equiv K_1$.

2. Preliminary lemmas

Lemma 2.1 ([3]). *Let* $P(z) = \frac{1+Cw(z)}{1+Dw(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n$, *then*

$$|p_n| \leq (C-D).$$

Result is sharp for the functions $P_n(z) = \frac{1+C\delta z^n}{1+D\delta z^n}$, $|\delta| = 1$ and $n \geq 1$.

Lemma 2.2 ([4]). *Let* $g \in S^*(A,B)$, *then, for* $A-(n-1)B \geq (n-2)$, $(n \geq 3)$,

$$|b_n| \leq \frac{1}{(n-1)!} \prod_{k=2}^{n} (A-(k-1)B).$$

Equality holds for the function $g_0(z)$ *defined by*

$$g_0(z) = z(1+B\delta z)^{(A-B)/B}, \quad |\delta| = 1.$$

Since $g(z) \in C(A,B)$ implies that $zg'(z) \in S^*(A,B)$, we have the following

Lemma 2.3. *Let* $g \in C(A,B)$, *then, for* $A-(n-1)B \geq (n-2)$, $(n \geq 3)$,

$$|b_n| \leq \frac{1}{n!} \prod_{k=2}^{n} (A-(k-1)B).$$

Result is sharp for the function $g_1(z)$ *defined by*

$$g_1'(z) = (1+B\delta z)^{(A-B)/B}, \quad |\delta| = 1.$$

Lemma 2.4 ([5]). *Let* $g \in S^*(A,B)$, *then, for* $|s| \leq 1$, $|t| \leq 1$, $(s \neq t)$

$$\frac{tg(sz)}{sg(tz)} \prec \begin{cases} \left(\dfrac{1+Bsz}{1+Btz}\right)^{(A-B)/B}, & B \neq 0; \\ \exp A(s-t)z, & B = 0. \end{cases}$$

Lemma 2.5. *If* $g \in S^*(A,B)$, *then, for* $|z| = r < 1$,

$$r(1-Br)^{(A-B)/B} \leq |g(z)| \leq r(1+Br)^{(A-B)/B}, \quad B \neq 0; \tag{2.1}$$

$$r\exp(-Ar) \leq |g(z)| \leq r\exp(Ar), \quad B = 0; \tag{2.2}$$

$$\left|\arg \frac{g(z)}{z}\right| \leq \frac{(A-B)}{B} \sin^{-1}(Br), \quad B \neq 0; \tag{2.3}$$

$$\left|\arg\frac{g(z)}{z}\right| \le Ar, \quad B=0. \tag{2.4}$$

Equality sign in these bounds is attained by the function $g_0(z)$ defined by

$$g_0(z)=\begin{cases} z(1+B\delta z)^{(A-B)/B}, & B\neq 0; \\ z\exp(A\delta z), & B=0,\ |\delta|=1. \end{cases}$$

Proof. Letting $s\to 1$ and $t\to 0$ in the Lemma 2.4, we obtain

$$\frac{g(z)}{z} \prec (1+Bz)^{(A-B)/B}, \quad B\neq 0; \tag{2.5}$$

$$\frac{g(z)}{z} \prec \exp(Az), \quad B=0. \tag{2.6}$$

(2.5) implies that

$$\frac{g(z)}{z} = (1+Bw(z))^{(A-B)/B}, \quad B\neq 0. \tag{2.7}$$

Case (i) $B>0$.

$$\begin{aligned}\left|(1+Bw(z))^{(A-B)/B}\right| &= \left|\exp\left\{\frac{(A-B)}{B}\log(1+Bw(z))\right\}\right| \\ &= \exp\left\{\frac{(A-B)}{B}\log|1+Bw(z)|\right\} \\ &= |1+Bw(z)|^{(A-B)/B} \\ &\le (1+Br)^{(A-B)/B}.\end{aligned}$$

Case (ii) $B<0$.

Let $B=-B'$, $B'>0$. Then

$$\begin{aligned}\left|(1+Bw(z))^{(A-B)/B}\right| &= \left|\left\{\left(1-B'w(z)\right)^{-1}\right\}^{(A-B)/B'}\right| \\ &= \left|\left(1-B'w(z)\right)^{-1}\right|^{(A-B)/B'} \\ &\le \left(\frac{1}{1-B'r}\right)^{(A-B)/B'} \\ &= (1+Br)^{(A-B)/B}.\end{aligned}$$

Combining the cases (i) and (ii), (2.1) follows from (2.7). Similarly, we get (2.2) from (2.6). Again from (2.5), we obtain (2.3) as follows

$$\left|\arg\frac{g(z)}{z}\right| \le \frac{(A-B)}{B}|\arg(1+Bw(z))| \le \frac{(A-B)}{B}\sin^{-1}(Br).$$

Similarly (2.4) directly follows from (2.6).

On the same lines we can prove the following

Lemma 2.6. *If $g \in C(A,B)$, then, for $|z| = r < 1$,*

$$\frac{1}{A}\{1-(1-Br)^{A/B}\} \le |g(z)| \le \frac{1}{A}\{(1+Br)^{A/B}-1\}, \quad B \ne 0;$$
$$\frac{1}{A}\{1-\exp(-Ar)\} \le |g(z)| \le \frac{1}{A}\{\exp(Ar)-1\}, \quad B = 0;$$
$$\left|\arg\frac{g(z)}{z}\right| \le \frac{A}{B}\sin^{-1}(Br), \quad B \ne 0;$$
$$\left|\arg\frac{g(z)}{z}\right| \le Ar, \quad B = 0.$$

Lemma 2.7 ([2])**.** *Let f and g are analytic functions and h be convex univalent function in E such that $f \prec h$ and $g \prec h$. Then $(1-\lambda)f + \lambda g \prec h$, $(0 \le \lambda \le 1)$.*

3. Coefficient estimates

Theorem 3.1. *Let $f \in K(A,B;C,D)$. Then, for $A-(n-1)B \ge (n-2)$, $(n \ge 3)$,*

$$|a_n| \le \frac{1}{n!}\sum_{k=2}^{n}\{A-(k-1)B\} + \frac{(C-D)}{n}\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\{A-(j-1)B\}\right). \tag{3.1}$$

Bound (3.1) is sharp.

Proof. By definition of $K(A,B;C,D)$,

$$\frac{zf'(z)}{g(z)} = \frac{1+Cw(z)}{1+Dw(z)} = P(z).$$

Expanding the series,

$$\begin{aligned}&(z+2a_2z^2+\cdots+na_nz^n+\cdots)\\ &\quad = (z+b_2z^2+\cdots+b_{n-1}z^{n-1}+b_nz^n+\cdots)(1+p_1z+p_2z^2+\cdots+p_{n-1}z^{n-1}+\cdots).\end{aligned} \tag{3.2}$$

Equating the coefficients of z^n in (3.2),

$$na_n = b_n + p_1b_{n-1} + p_2b_{n-2} + \cdots + p_{n-2}b_2 + p_{n-1}.$$

Applying triangular inequality and Lemma 2.1 , we get

$$n|a_n| \le |b_n| + (C-D)\left(1+\sum_{k=2}^{n-1}|b_k|\right). \tag{3.3}$$

Using Lemma 2.2 in (3.3), we obtain

$$n|a_n| \le \frac{1}{(n-1)!}\prod_{k=2}^{n}\{A-(k-1)B\} + (C-D)\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\{A-(j-1)B\}\right)$$

which yields (3.1). The bound (3.1) is sharp for the function $f_0(z)$ defined by

$$f_0(z) = \left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)(1+B\delta_2 z)^{(A-B)/B}, \quad |\delta_1| = |\delta_2| = 1.$$

Similarly we can prove

Theorem 3.2. *Let $f \in K_1(A,B;C,D)$. Then, for $A-(n-1)B \geq (n-2)$, $(n \geq 3)$,*

$$|a_n| \leq \frac{1}{n}\left[\frac{1}{n!}\prod_{k=2}^{n}(A-(k-1)B) + (C-D)\left(1+\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}\{A-(j-1)B\}\right)\right]. \tag{3.4}$$

The bound (3.4) is sharp for the function $f_1(z)$ given by

$$f_1'(z) = \frac{1}{Az}\left(\frac{1+C\delta_1 z}{1+D\delta_1 z}\right)\{(1+B\delta_2 z)^{A/B}-1\}, \quad |\delta_1| = |\delta_2| = 1.$$

Remark 3.1. (i) If $f \in K(1,-1;C,D) \equiv K(C,D)$, $|a_n| \leq 1 + \frac{(n-1)(C-D)}{2}$ which is the result due to Mehrok [10].

(ii) If $f \in K_1(1,-1;C,D) \equiv K_1(C,D)$, $|a_n| \leq \frac{1}{n}\{1+(n-1)(C-D)\}$, a result due to Mehrok and Singh [11].

Remark 3.2. (i) If $f \in K(A,B;1,-1) \equiv K^*(A,B)$, for $A-(n-1)B \geq (n-2)$, $(n \geq 3)$,

$$|a_n| \leq \frac{1}{n!}\prod_{k=2}^{n}\{A-(k-1)B\} + \frac{2}{n}\left(1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}\{A-(j-1)B\}\right).$$

This result was proved by Goel and Mehrok [5].

(ii) If $f \in K_1(A,B;1,-1) \equiv K_1^*(A,B)$, for $A-(n-1)B \geq (n-2)$, $(n \geq 3)$,

$$|a_n| \leq \frac{1}{n(n!)}\prod_{k=2}^{n}\{A-(k-1)B\} + \frac{2}{n}\left(1+\sum_{k=2}^{n-1}\frac{1}{k!}\prod_{j=2}^{k}\{A-(j-1)B\}\right).$$

Remark 3.3. (i) If $f \in K(1,-1;1,-1) \equiv K$, then $|a_n| \leq n$. The result due to Reade [13].

(ii) If $f \in K_1(1,-1;1,-1) \equiv K_1$, then $|a_n| \leq 2 - \frac{1}{n}$. This result was obtained by Silverma and Telage [15].

4. Distortion theorems

Theorem 4.1. *Let $f \in K(A,B;C,D)$, then*

$$\left(\frac{1-Cr}{1-Dr}\right)(1-Br)^{(A-B)/B} \leq |f'(z)| \leq \left(\frac{1+Cr}{1+Dr}\right)(1+Br)^{(A-B)/B}, \quad B \neq 0; \tag{4.1}$$

$$\left(\frac{1-Cr}{1-Dr}\right)\exp(-Ar) \leq |f'(z)| \leq \left(\frac{1+Cr}{1+Dr}\right)\exp(Ar), \quad B = 0; \tag{4.2}$$

$$\int_0^r \left(\frac{1-Cu}{1-Du}\right)(1-Bu)^{(A-B)/B}du \le |f(z)| \le \int_0^r \left(\frac{1+Cu}{1+Du}\right)(1+Bu)^{(A-B)/B}du, \quad B \neq 0; \tag{4.3}$$

$$\int_0^r \left(\frac{1-Cu}{1-Du}\right)\exp(-Au)du \le |f(z)| \le \int_0^r \left(\frac{1+Cu}{1+Du}\right)\exp(Au)du, \quad B = 0. \tag{4.4}$$

All these bounds are sharp.

Proof. Since $f \in K(A,B;C,D)$, it follows that

$$\frac{zf'(z)}{g(z)} = \frac{1+Cw(z)}{1+Dw(z)}$$

which maps $|w(z)| \le r$ onto the circle

$$\left|\frac{(zf'(z)}{g(z)} - \frac{1-CDr^2}{1-D^2r^2}\right| \le \frac{(C-D)r}{(1-D^2r^2)}.$$

This yields

$$\frac{(1-Cr)}{(1-Dr)} \le \left|\frac{(zf'(z)}{g(z)}\right| \le \frac{(1+Cr)}{(1+Dr)}$$

which further implies that

$$\frac{(1-Cr)}{(1-Dr)}|g(z)| \le |zf'(z)| \le \frac{(1+Cr)}{(1+Dr)}|g(z)|. \tag{4.5}$$

Using (2.1) and (2.2) along with (4.5), we obtain (4.1) and (4.2).

Now

$$\begin{aligned} |f(z)| &= \int_0^z f'(z)dz \\ &\le \int_0^r |f'(z)|dr \\ &\le \begin{cases} \displaystyle\int_0^r \left(\frac{1+Cr}{1+Dr}\right)(1+Br)^{(A-B)/B}dr, & B \neq 0; \\ \displaystyle\int_0^r \left(\frac{1+Cr}{1+Dr}\right)\exp(Ar)dr, & B = 0. \end{cases} \end{aligned}$$

Let z_0, $|z_0| = 1$, be so chosen that $|f(z_0)| \le |f(z)|$ for all z, $|z| = r$. If $L(z_0)$ is the pre-image of the segment $\overline{[0, f(z_0)]}$ in E, then

$$\begin{aligned} |f(z)| &= \int_{L(z_0)} |f'(z)|dz \\ &\ge \begin{cases} \displaystyle\int_0^r \left(\frac{1-Cr}{1-Dr}\right)(1-Br)^{(A-B)/B}dr, & B \neq 0; \\ \displaystyle\int_0^r \left(\frac{1-Cr}{1-Dr}\right)\exp(-Ar)dr, & B = 0. \end{cases} \end{aligned}$$

Equality signs in (4.1), (4.2), (4.3) and (4.4) are attained by the function $f_2(z)$ defined by

$$f_2'(z)=\begin{cases}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)(1+B\delta_2 r)^{(A-B)/B}, & B\neq 0;\\ \left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)\exp(A\delta_2 z), & B=0,\ |\delta_1|=|\delta_2|=1.\end{cases}\tag{4.6}$$

Similarly, by using Lemma 2.6, we can prove

Theorem 4.2. *Let $f\in K_1(A,B;C,D)$, then*

$$\frac{1}{Ar}\left(\frac{1-Cr}{1-Dr}\right)\{1-(1-Br)^{A/B}\}\le|f'(z)|\le\frac{1}{Ar}\left(\frac{1+Cr}{1+Dr}\right)\{(1+Br)^{A/B}-1\},\quad B\neq 0;$$

$$\frac{1}{Ar}\left(\frac{1-Cr}{1-Dr}\right)\exp(-Ar)\le|f'(z)|\le\frac{1}{Ar}\left(\frac{1+Cr}{1+Dr}\right)\exp(Ar),\quad B=0;$$

$$\frac{1}{A}\int_0^r\frac{1}{u}\left(\frac{1-Cu}{1-Du}\right)\{1-(1-Bu)^{A/B}\}du\le|f(z)|\le\frac{1}{A}\int_0^r\frac{1}{u}\left(\frac{1+Cr}{1+Dr}\right)\{(1+Bu)^{A/B}-1\}du,\quad B\neq 0;$$

$$\frac{1}{A}\int_0^r\frac{1}{u}\left(\frac{1-Cu}{1-Du}\right)\exp(-Au)du\le|f(z)|\le\frac{1}{A}\int_0^r\frac{1}{u}\left(\frac{1+Cu}{1+Du}\right)\exp(Au)du,\quad B=0.$$

All these bounds are sharp and extremal function is given by $f_3(z)$ defined by

$$f_3'(z)=\begin{cases}\dfrac{1}{Az}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)\{(1+B\delta_2 z)^{A/B}-1\}, & B\neq 0;\\ \dfrac{1}{Az}\left(\dfrac{1+C\delta_1 z}{1+D\delta_1 z}\right)\exp(A\delta_2 z), & B=0,\ |\delta_1|=|\delta_2|=1.\end{cases}\tag{4.7}$$

5. Argument theorems

Theorem 5.1. *Let $f\in K(A,B;C,D)$, then*

$$|\arg f'(z)|\le\frac{(A-B)}{B}\sin^{-1}(Br)+\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\},\quad B\neq 0;\tag{5.1}$$

$$|\arg f'(z)|\le Ar+\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\},\quad B=0.\tag{5.2}$$

The results are sharp.

Proof. From (1.11), we have $\frac{zf'(z)}{g(z)}=\frac{1+Cw(z)}{1+Dw(z)}$. Since the transformation $\frac{zf'(z)}{g(z)}=\frac{1+Cw(z)}{1+Dw(z)}$ maps $|w(z)|\le r$ onto the circle

$$\left|\frac{zf'(z)}{g(z)}-\frac{1-CDr^2}{1-D^2r^2}\right|\le\frac{(C-D)r}{(1-D^2r^2)},$$

therefore

$$\left|\arg\frac{zf'(z)}{g(z)}\right|\le\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}.$$

This implies that

$$|\arg f'(z)|\le\left|\arg\frac{g(z)}{z}\right|+\sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}.\tag{5.3}$$

Inequality (5.3) together with (2.3) and (2.4) yield (5.1) and (5.2) respectively. Equality sign in (5.1) and (5.2) holds for the function $f_2(z)$ defined by (4.6) in which

$$\delta_1 = \frac{r}{z}\left[\frac{-(C+D)r + i\{(1-C^2r^2)(1-D^2r^2)\}^{1/2}}{(1+CDr^2)}\right] \tag{5.4}$$

and

$$\delta_1 = \frac{r}{z}\{-Br + i(1-B^2r^2)^{1/2}\}. \tag{5.5}$$

Similarly, by using Lemma 2.6, we have

Theorem 5.2. *Let $f \in K_1(A,B;C,D)$, then*

$$|\arg f'(z)| \le \frac{A}{B}\sin^{-1}(Br) + \sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}, \quad B \ne 0;$$
$$|\arg f'(z)| \le Ar + \sin^{-1}\left\{\frac{(C-D)r}{(1-CDr^2)}\right\}, \quad B \ne 0.$$

The results are sharp for the function $f_3(z)$ defined in (4.7) where δ_1 and δ_2 are given by (5.4) and (5.5), respectively.

Remark 5.4. Taking $A = 1$ and $B = -1$ in the Theorem 4.1, we get the result proved by Mehrok [10].

Remark 5.5. On taking $C = 1$ and $D = -1$ in the Theorems 4.1 and 5.1, we get the results due to Goel and Mehrok [5].

Remark 5.6. Letting $A = C = 1$ and $B = D = -1$ in the Theorems 4.1 and 5.1, we obtain the results proved by Ogawa [12] and Krzyz [8] for the class K.

Remark 5.7. For $C = 1$ and $D = -1$ in Theorems 4.2 and 5.2, we get the results established by Gawad and Thomas [1].

6. Convex set of functions

Theorem 6.1. *If f and $h \in K(A,B;C,D)$, then*

$$(1-\lambda)f + \lambda h \in K(A,B;C,D), \quad (0 \le \lambda \le 1).$$

Proof. Since $\frac{1+Cz}{1+Dz}$ is convex univalent in E, the theorem follows by Lemma 2.7 definition of $K(A,B;C,D)$.

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