MULTI-PARAMETER BURAU REPRESENTATIONS

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Abstract. We consider the multi-parameter representation of Artin's braid group introduced by D. D. Long and J. P. Tian, namely $\alpha : B_n \to GL_m(C)$, where $m = n!n$. First, we show that there exists a complex specialization of the multi-parameter representation that does not arise from any Hecke algebra. Second, we find conditions under which the images of the generators of the braid group on three strings under the multi-parameter representation are unitary relative to a nonsingular hermitian matrix.

1. Introduction

The braid group, B_n, has a well- known representation due to Artin in the group $\mathrm{Aut}(F_n)$ of automorphisms of the free group F_n generated by $x_1, \ldots, x_n$. The automorphism corresponding to the braid generator σ_i takes x_i to $x_i x_{i+1} x_i^{-1}$; x_{i+1} to x_i, and fixes all other free generators. Applying the Fox derivatives and Magnus representation, we get the classical Burau representation which is of degree n. In [3], D.D. Long gave a new derivative of the Burau representation by using an action of $\mathrm{Aut}(F_n)$ on the representation variety $R = R(F_n, G)$, where G is a compact semisimple Lie group and F_n is the free group of rank n.

In Section 2, we define the multi-parameter representation, namely $\alpha : B_n \to GL_m(\mathbb{C})$, where $m = n!n$. Explicit matrices will be computed in the case $n = 3$. For more details, see [6]. In Section 3, we prove that there exist complex specializations of the multi-parameter representation that do not arise from any Hecke algebra. In Section 4, we determine sufficient conditions under which the complex specializations of the multi-parameter representation of the braid group on three strings is unitary relative to a nonsingular hermitian matrix .

2. Definitions

Definition 1 ([2])**.** The braid group on n strings, B_n, is the abstract group with presentation $B_n = \{\sigma_1, \ldots, \sigma_{n-1} / \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, 2, \ldots, n-2, \sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1\}$.

The generators $\sigma_1, \ldots, \sigma_{n-1}$ are called the standard generators of B_n.

Definition 2. [2] The pure braid group, denoted by P_n, is defined as the kernel of the homomorphism $B_n \to S_n$ defined by $\sigma_i \to (i, i+1)$, $1 \le i \le n-1$. It is finitely generated by the elements

$$A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}, 1 \le i < j \le n.$$

Definition 3. Let G be a finitely generated group. A representation $G \to GL_r(C)$ is reducible if there exists a non-zero proper subspace of C^r that is invariant under the action of the generators of G.

Applying the new derivative of the Burau representation adopted by D. D. Long in [3], J.P. Tian has made some computations by considering the action of the braid group B_n on a special case of a representation variety, namely $R = R(F_n, G)$, where $G = SU(2, \mathbb{C})$. This gives us the definition of the multi-parameter representation.

Let us consider V_t as a linear space which is a summation of all tangent spaces of R at its base points. $\beta \in Aut(F_n)$ acts on R and $\beta^*(p_\theta^t) = p_\theta^{\tau_\beta(t)}$, where τ_β is a permutation determined by β and acting on t in an obvious way.

Definition 4. [6] The multi-parameter representation of the braid group is a map $\alpha : B_n \to GL(V_t)$, where $\alpha(\beta) = \oplus_{\tau \in S_n} d\beta^*_{p_{\theta^{\tau(t)}}}$. Here $\tau(t)$ is a permutation of $(t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$, $\beta^* \in Diff(R)$ and $p_{\theta^{\tau(t)}}$ are the base points.

For a particular permutation $t = (t_1, t_2, \cdots, t_n)$, we compute the matrix representations of the images of the generators of B_n under the multi-parameter representation, namely $\alpha(\sigma_k)$. The order we give for the permutations of $(t_1, t_2, \cdots, t_n)$ is the lexicographic order.

For a fixed choice $t = (\ldots, \underbrace{t_i}_{k}, t_j, \ldots)$, we have that

$$d(\sigma_k^*)_{p_\theta^t} = M_k(t_i, t_j) = I_{k-1} \oplus \begin{pmatrix} 1 - \lambda^{t_j} & \lambda^{t_i} \\ 1 & 0 \end{pmatrix} \oplus I_{n-k-1}.$$

Here $k = 1, \cdots, n-1$ and $1 \le i \ne j \le n$. We then compute $d(\sigma_k^*)_{p_\theta^{\tau(t)}}$, where τ is a permutation in S_n. We expect to have $n!$ square matrices $M_k(t_i, t_j)$, each of size $n \times n$. In order to place these square $n \times n$ matrices $M_k(t_i, t_j)$ in the matrix $\alpha(\sigma_k)$, we simply order the n-tuples in the lexicographic order. In the case $n = 3$, the tuples are ordered as follows: (t_1, t_2, t_3), (t_1, t_3, t_2), (t_2, t_1, t_3), (t_2, t_3, t_1), (t_3, t_1, t_2) and (t_3, t_2, t_1). More precisely, the images of the generators of B_3 under the multi-parameter representation $\alpha : B_3 \to GL_{18}(\mathbb{C})$ are given by

$$\alpha(\sigma_1) = \begin{pmatrix} 0 & 0 & M_1(t_2, t_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1(t_3, t_1) & 0 \\ M_1(t_1, t_2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_1(t_3, t_2) \\ 0 & M_1(t_1, t_3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1(t_2, t_3) & 0 & 0 \end{pmatrix}$$

and

$$\alpha(\sigma_2) = \begin{pmatrix} 0 & M_2(t_3, t_2) & 0 & 0 & 0 & 0 \\ M_2(t_2, t_3) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2(t_3, t_1) & 0 & 0 \\ 0 & 0 & M_2(t_1, t_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_2(t_2, t_1) \\ 0 & 0 & 0 & 0 & M_2(t_1, t_2) & 0 \end{pmatrix}.$$

The matrices $M_k(t_i, t_j)$, $k = 1, 2$ and $1 \le i \ne j \le 3$, are given by

$$M_1(t_i, t_j) = \begin{pmatrix} 1 - z_j & z_i & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2(t_i, t_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - z_j & z_i \\ 0 & 1 & 0 \end{pmatrix}.$$

Here $z_i = \lambda^{t_i}$ and $z_j = \lambda^{t_j}$. In other words, the first column of $\alpha(\sigma_1)$ corresponds to $d(\sigma_1^*)_{p_\theta^t}$, where $t = (t_1, t_2, t_3)$. We observe that if $X \in T_{p_\theta^t}(R) \subset V_t$ then $d(\sigma_1^*)_{p_\theta^t} \in T_{p_\theta^{\sigma_1(t)}}(R) = T_{p_\theta^{(t_2, t_1, t_3)}}(R)$. Since the tuples are arranged in the lexicographic order, it follows that the matrix $M_1(t_1, t_2)$ comes third in the first column. The same is true for all the other square matrices $M_k(t_i, t_j)$.

3. Multi-Parameter Representation and Hecke Algebra

The Hecke algebra is an algebra defined by generators $g_1, \ldots, g_{n-1}$ with relations

$$g_i g_j = g_j g_i \quad |i - j| \ge 2,$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad 1 \le i \le n - 2,$$

$$g_i^2 = (q - 1)g_i + q \quad 1 \le i \le n - 1.$$

Here q ia a parameter.

In this section, we show that the multi-parameter representation does not arise from any Hecke algebra.

Lemma 5. *The number of eigenvalues of a $k \times k$ matrix A is at least the number of eigenvalues of the matrix A^2.*

Proof. Let x be an eigenvalue of A^2 then $\text{Det}(A^2 - xI_k) = 0$. Since $A^2 - xI_k = (A - \sqrt{x}I_k)(A + \sqrt{x}I_k)$, it follows that $\text{Det}((A - \sqrt{x}I_k)(A + \sqrt{x}I_k)) = 0$ which is equivalent to $\text{Det}((A - \sqrt{x}I_k)) = 0$ or $\text{Det}((A + \sqrt{x}I_k)) = 0$. This implies that $\sqrt{x}$ or $-\sqrt{x}$ (or both) are eigenvalues of A. $\square$

We now prove a result concerning the pure braid group, which will be used later to show that certain complex specializations of the multi-parameter representation do not arise from any Hecke algebra.

Lemma 6. *If there exist integers k, l, r, $s \in \{1, \ldots, n\}$ such that $z_k z_l \neq z_r z_s$ and neither one of them equals 1 then $\alpha : P_n \rightarrow GL_m(C)$ has more than 2 distinct eigenvalues. ($m = n!n$ and $n \geq 3$).*

Proof. For $1 \leq i \neq j \leq n$, we consider $N_{ij} = M_1(t_i, t_j) M_1(t_j, t_i)$, $1 \leq i \neq j \leq n$. The matrix corresponding to N_{ij} is

$$N_{ij} = \begin{pmatrix} 1 - z_j + z_i z_j & z_j(1 - z_j) \\ 1 - z_i & z_j \end{pmatrix} \oplus I_{n-2}.$$

It is easy to see that $A_{12} = \sigma_1^2$ is direct sum of N_{ij}, where each block N_{ij} occurs $(n-2)!$ times. The characteristic equation of A_{12} is $\text{Det}(A_{12} - xI_m) = 0$, which is equivalent to

$$\prod_{1 \leq i \neq j \leq n} \text{Det}(N_{ij} - xI_n) = \prod_{1 \leq i < j \leq n} (\text{Det}(N_{ij} - xI_n))^2 = 0.$$

Since $\text{Det}(N_{ij} - xI_n) = (1 - x)^{n-1}(z_i z_j - x)$, it follows that the characteristic equation of A_{12} is

$$(1 - x)^{n!(n-1)} \prod_{1 \leq i < j \leq n} (z_i z_j - x)^{(n-2)!2}.$$

We have that 1 and $z_i z_j$ are eigenvalues of A_{12} for every $1 \leq i \neq j \leq n$. Here, $z_i z_j$ has multiplicity $(n-2)!2$ because $\text{Det}(N_{ij} - xI_n) = \text{Det}(N_{ji} - xI_n) = (1 - x)^{n-1}(z_i z_j - x)$ and the fact that N_{ij} occurs $(n-2)!$ times in the matrix corresponding to A_{12}.

It is easy to see, by our hypothesis, that 1, $z_k z_l$ and $z_r z_s$ are distinct eigenvalues of A_{12}.

Direct computations show that $(N_{ij} - I_n)(N_{ij} - z_r z_s I_n) = (z_i z_j - z_r z_s)(N_{ij} - I_n)$, $1 \leq r < s \leq n$ and $(N_{ij} - I_n)(N_{ij} - z_i z_j I_n)$ is the $n \times n$ zero matrix. This implies that the minimal polynomial of A_{12} is $(x - 1) \prod_{1 \leq i < j \leq n} (x - z_i z_j)$. $\square$

Theorem 7. *If there exist integers k, l, r, $s \in \{1, \cdots, n\}$ such that $z_k z_l \neq z_r z_s$ and neither one of them equals 1 then $\alpha : B_n \rightarrow GL_m(C)$ does not arise from any Hecke algebra. ($m = n!n$ and $n \geq 2$).*

Proof. We consider the following cases:

Case $n = 2$: The matrix corresponding to σ_1 is $\alpha(\sigma_1) = \begin{pmatrix} 0 & 0 & 1 - z_1 & z_2 \\ 0 & 0 & 1 & 0 \\ 1 - z_2 & z_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

The minimal polynomial of $\alpha(\sigma_1)$ is $(x^2 - 1)(x^2 - z_1 z_2)$. It is clear that $\alpha(\sigma_1)$ has 4 distinct eigenvalues namely, $-1, 1, -\sqrt{z_1 z_2}$ and $\sqrt{z_1 z_2}$.

Case $n > 2$: Using our hypothesis, Lemma 6 asserts that the representation restricted to P_n has more than 2 distinct eigenvalues. We then conclude, by Lemma 5, that the representation α defined on the braid group B_n has more than 2 distinct eigenvalues.

Since the generators of the Hecke algebra satisfy a quadratic equation, it follows that it has at most 2 distinct eigenvalues and so α does not arise from any Hecke algebra. $\square$

4. Multi-parameter representation is unitary

Notation 8. *Let $(*) : M_m(\mathbb{C}[t^{\pm 1}])$ be an involution defined as follows:*

$$(f_{ij}(t))^* = f_{ji}(t^{-1}), f_{ij}(t) \in \mathbb{C}[t^{\pm 1}].$$

Definition 9. Let X and U be elements of $GL_m(\mathbb{C})$ where $m = n!n$. U is called a unitary element (relative to X) if $UXU^* = X$.

Since the matrices corresponding to the generators of B_n under the multi-parameter representation are large and of degree $n!n$, we rather perform our computations only in the case $n = 3$. First, we show that the complex specialization of the multi-parameter representation is reducible. Then, we determine sufficient conditions under which the complex specializations of the reducible multi-parameter representation is unitary relative to some invertible hermitian matrix. Since the representation is reducible, it follows that the choice of the hermitian matrix is not unique.

Theorem 10. *The multi-parameter representation $\alpha : B_3 \to GL_{18}(C)$ is reducible.*

Proof. We consider two cases for z_i.

If $z_i = z_j$ for all $i, j \in \{1, 2, 3\}$ then we consider the invariant subspace generated by the column vector $(1, \dots, 1)^T$, where T is the transpose.

If there exist $i \neq j$ such that $z_i \neq z_j$ then we consider the invariant subspace generated by the column vector $(v_1, v_2, v_3, v_4, v_5, v_6)^T$, where $v_1 = (z_1 - 1, z_2 - 1, z_3 - 1)^T$, $v_2 = (z_1 - 1, z_3 - 1, z_2 - 1)^T$, $v_3 = (z_2 - 1, z_1 - 1, z_3 - 1)^T$, $v_4 = (z_2 - 1, z_3 - 1, z_1 - 1)^T$, $v_5 = (z_3 - 1, z_1 - 1, z_2 - 1)^T$ and $v_6 = (z_3 - 1, z_2 - 1, z_1 - 1)^T$. $\square$

Theorem 11. *The multi-parameter representation $\alpha : B_3 \to GL_{18}(\mathbb{C})$ is unitary under any of the following conditions:*

(i) $z_1 \neq z_2$, $z_1^2 z_3 \neq 1$ and $z_i \neq 1$ ($i = 1, 2, 3$).

(ii) $z_1 = z_2 = z_3$.

(iii) $z_1 = z_2 = 1$ and $z_3 \neq 1$.

(iv) $z_1 = z_3 = 1$ and $z_2 \neq 1$.

(v) $z_2 = z_3 = 1$ and $z_1 \neq 1$.

Proof. (i) $z_1 \neq z_2$, $z_1^2 z_3 \neq 1$ and $z_i \neq 1$ ($i = 1, 2, 3$). Let $H = \oplus_{i=1}^{6} H_i$, where

$$H_1 = \begin{pmatrix} \frac{z_1^2 - z_2}{z_1(-1+z_2)} & 1 & \frac{z_2(-1+z_3)}{z_3(-1+z_2)} \\ 1 & 1 & \frac{z_2(-1+z_3)}{z_3(-1+z_1)} \\ \frac{-1+z_3}{-1+z_2} & \frac{(-1+z_3)z_1}{(-1+z_1)z_2} & \frac{(-1+z_3)(-z_2+z_1 z_3)}{(-1+z_1)(-1+z_2)z_3} \end{pmatrix}, H_2 = \begin{pmatrix} \frac{z_1^2 - z_2}{z_1(-1+z_2)} & \frac{z_2(-1+z_3)}{z_3(-1+z_2)} & 1 \\ \frac{-1+z_3}{-1+z_2} & \frac{(-1+z_3)(-z_2+z_1 z_3)}{(-1+z_1)(-1+z_2)z_3} & \frac{(-1+z_3)}{(-1+z_1)} \\ 1 & \frac{(-1+z_3)z_1}{(-1+z_1)z_3} & 1 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 1 & \frac{z_2}{z_1} & \frac{z_2(-1+z_3)}{z_3(-1+z_1)} \\ \frac{z_1}{z_2} & \frac{z_1^2 - z_2}{z_1(-1+z_2)} & \frac{z_2(-1+z_3)}{z_3(-1+z_2)} \\ \frac{z_1(-1+z_3)}{z_2(-1+z_1)} & \frac{(-1+z_3)}{(-1+z_2)} & \frac{(-1+z_3)(-z_2+z_1 z_3)}{(-1+z_1)(-1+z_2)z_3} \end{pmatrix}, H_4 = \begin{pmatrix} 1 & \frac{z_2(-1+z_3)}{z_3(-1+z_1)} & \frac{z_2}{z_1} \\ \frac{(-1+z_3)z_1}{(-1+z_1)z_2} & \frac{(-1+z_3)(-z_2+z_1 z_3)}{(-1+z_1)(-1+z_2)z_3} & \frac{z_2(-1+z_3)}{z_1(-1+z_2)} \\ \frac{z_1}{z_2} & \frac{(-1+z_3)z_1}{(-1+z_2)z_3} & \frac{z_1^2 - z_2}{z_1(-1+z_2)} \end{pmatrix},$$

$$H_5 = \begin{pmatrix} \frac{(-1+z_3)(-z_2+z_1 z_3)}{(-1+z_1)(-1+z_2)z_3} & \frac{z_2(-1+z_3)}{z_1(-1+z_2)} & \frac{(-1+z_3)}{(-1+z_1)} \\ \frac{z_1(-1+z_3)}{z_3(-1+z_2)} & \frac{z_1^2 - z_2}{z_1(-1+z_2)} & 1 \\ \frac{z_1(-1+z_3)}{z_3(-1+z_1)} & 1 & 1 \end{pmatrix} \text{ and } H_6 = \begin{pmatrix} \frac{(-1+z_3)(-z_2+z_1 z_3)}{(-1+z_1)(-1+z_2)z_3} & \frac{(-1+z_3)}{(-1+z_1)} & \frac{z_2(-1+z_3)}{z_1(-1+z_2)} \\ \frac{z_1(-1+z_3)}{z_3(-1+z_1)} & 1 & \frac{z_2}{z_1} \\ \frac{z_1(-1+z_3)}{z_3(-1+z_2)} & \frac{z_1}{z_2} & \frac{z_1^2 - z_2}{z_1(-1+z_2)} \end{pmatrix}.$$

It is clear that H_i is hermitian for every $1 \leq i \leq 6$. We also have that $\mathrm{Det}(H_i) = \frac{(-1+z_3)(-1+z_3 z_1^2)(z_1-z_2)^2}{(-1+z_1)^2(-1+z_2)^2 z_3 z_1} \neq 0$ by our hypothesis. This implies that H is hermitian and invertible. Also, we have that

$$\sigma_1 H \sigma_1^* = M_1(t_2, t_1) H_3 (M_1(t_2, t_1))^* \oplus M_1(t_3, t_1) H_5 (M_1(t_3, t_1))^* \oplus M_1(t_1, t_2) H_1 (M_1(t_1, t_2))^* \oplus$$
$$M_1(t_3, t_2) H_6 (M_1(t_3, t_2))^* \oplus M_1(t_1, t_3) H_2 (M_1(t_1, t_3))^* \oplus M_1(t_2, t_3) H_4 (M_1(t_2, t_3))^*$$

and

$$\sigma_2 H \sigma_2^* = M_2(t_3, t_2) H_2 (M_2(t_3, t_2))^* \oplus M_2(t_2, t_3) H_1 (M_2(t_2, t_3))^* \oplus M_2(t_3, t_1) H_4 (M_2(t_3, t_1))^* \oplus$$
$$M_2(t_1, t_3) H_3 (M_2(t_1, t_3))^* \oplus M_2(t_2, t_1) H_6 (M_2(t_2, t_1))^* \oplus M_2(t_1, t_2) H_5 (M_2(t_1, t_2))^*.$$

Direct computations show that $\sigma_1 H \sigma_1^* = \sigma_2 H \sigma_2^* = H$.

(ii) $z_1 = z_2 = z_3 = z$. We consider three cases, $z = 1$, $z = -1$ and $z \neq \pm 1$.

Case $z = 1$: $\sigma_1\sigma_1^* = \sigma_2\sigma_2^* = I_{18}$.

Case $z = -1$: $M_1(t_r, t_s) = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M_2(t_r, t_s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, where $1 \le r \ne s \le 3$.

Let $H = \oplus_1^6 M$ where $M = \begin{pmatrix} 1 & 1+i & 1+i \\ 1-i & 1 & 1+i \\ 1-i & 1-i & 1 \end{pmatrix}$. Here i is a complex number such that $i^2 = -1$.

Since $M_1(t_r, t_s)M((M_1(t_r, t_s))^* = M_2(t_r, t_s)M((M_2(t_r, t_s))^* = M$, it follows that $\sigma_1 H \sigma_1^* = \sigma_2 H \sigma_2^* = H$. Note that H is hermitian and invertible whose determinant is equal to one.

Case $z \ne \pm 1$. $M_1(t_r, t_s) = \begin{pmatrix} 1-z & z & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M_2(t_r, t_s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-z & z \\ 0 & 1 & 0 \end{pmatrix}$, where $1 \le r \ne s \le 3$.

Let $H = \oplus_1^6 M$, where $M = \begin{pmatrix} 0 & 1+z & 1+z \\ \frac{1+z}{z} & 0 & 1+z \\ \frac{1+z}{z} & \frac{1+z}{z} & 0 \end{pmatrix}$. It is easy to see that $M_1(t_r, t_s)M(M_1(t_r, t_s))^* = M$ and $M_2(t_r, t_s)M(M_2(t_r, t_s))^* = M$. Therefore, we get that $\sigma_1 H \sigma_1^* = \sigma_2 H \sigma_2^* = H$, where H is an invertible hermitian matrix whose determinant is equal to $\frac{(z+1)^{24}}{z^{12}}$. It is worth to mention that $M_k(t_i, t_j)$ is the Burau representation ($k = 1, 2$ and $1 \le i \ne j \le 3$).

(iii) $z_1 = z_2 = 1$ and $z_3 \ne 1$. Let $H = \oplus_{i=1}^6 H_i$, where

$$H_3 = H_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, H_4 = H_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -z_3 \\ 0 & \frac{-1}{z_3} & 1 \end{pmatrix} \text{ and } H_6 = H_5 = \begin{pmatrix} 1 & -z_3 & -z_3 \\ \frac{-1}{z_3} & 1 & 0 \\ \frac{-1}{z_3} & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\sigma_1 H \sigma_1^* = \sigma_2 H \sigma_2^* = H$, where H is an invertible hermitian matrix whose determinant is equal to one.

(iv) $z_1 = z_3 = 1$ and $z_2 \ne 1$. Let $H = \oplus_{i=1}^6 H_i$, where

$$H_6 = H_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -z_2 \\ 0 & \frac{-1}{z_2} & 1 \end{pmatrix}, H_5 = H_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \text{ and } H_4 = H_3 = \begin{pmatrix} 1 & -z_2 & -z_2 \\ \frac{-1}{z_2} & 1 & 0 \\ \frac{-1}{z_2} & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\sigma_1 H \sigma_1^* = \sigma_2 H \sigma_2^* = H$, where H is an invertible hermitian matrix whose determinant is equal to one.

(v) $z_2 = z_3 = 1$ and $z_1 \neq 1$. Let $H = \oplus_{i=1}^{6} H_i$, where

$$H_2 = H_1 = \begin{pmatrix} 1 & z_1 & z_1 \\ \frac{1}{z_1} & -1 & 0 \\ \frac{1}{z_1} & 0 & -1 \end{pmatrix}, H_5 = H_3 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & z_1 \\ 0 & \frac{1}{z_1} & -1 \end{pmatrix} \text{ and } H_4 = H_6 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

It is easy to see that $\sigma_1 H \sigma_1^* = \sigma_2 H \sigma_2^* = H$, where H is an invertible hermitian matrix whose determinant is equal to 3^6. □

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