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THE ISOMORPHISMS AND THE CENTER OF WEAK QUANTUM ALGEBRAS $\omega sl_q(2)$

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Abstract. The aim of this paper is to describe the centre as well as the structure of $\omega sl_q(2)$ by applying the deformation of Harish-Chandra homomorphism.

Introduction

Throughout, the basic field is the complex number field $\mathbb{C}$. All algebras, modules and vector spaces are over $\mathbb{C}$ unless otherwise specified. Let q be a parameter which is not a root of unity.

F. Li and S. Duplij [9] constructed a quantum algebra $\omega sl_q(2)$. By definition, the quantum algebra $\omega sl_q(2)$ is generated by the four variables $E, F, K, \overline{K}$ with the relations:

$$K\overline{K} = \overline{K}K = J \tag{1}$$

$$JK = K, \overline{K}J = \overline{K} \tag{2}$$

$$KE = q^2 EK,\ \overline{K}E = q^{-2}E\overline{K} \tag{3}$$

$$KF = q^{-2}FK,\ \overline{K}F = q^2 F\overline{K} \tag{4}$$

$$EF - FE = \frac{K - \overline{K}}{q - q^{-1}} \tag{5}$$

This is an interesting example of weak Hopf algebras in the sense of [7]. In the paper [9], the authors gave a detail description of the structure theory of $\omega sl_q(2)$, such as its basis, group-like elements, regular quasi-R matrix and so on.

As a continuation of the paper [9], we will study the isomorphisms among these weak quantum algebras and their centre. Several people have considered the problems of Hopf algebra automorphisms. For example, [1, 4, 11]. In [3] the isomorphisms among quantum algebras $U_{r,s}(sl_n)$ with different parameters r, s were investigated. However, nobody has considered the same problem for the weak quantum algebra $\omega sl_p(2)$. By applying the idea of [3] and some known facts, we can yield the group of automorphisms of weak Hopf algebra $\omega sl_q(2)$. It is shown that $\varphi : \omega sl_q(2) \to \omega sl_p(2)$ is a weak Hopf

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algebra isomorphism if and only if $p = \pm\, q$. If this is the case, we will determine all such isomorphisms. Let $U_q(sl_2)$ be the quantum group corresponding to three dimensional semisimple Lie algebra sl_2. As is known, one of many beautiful results for $U_q(sl_2)$ is that the centre of $U_q(sl_2)$ can be described by the Harish-Chandra homomorphism (see [6]). Similar to the case of $U_q(sl_2)$, we would like to study the centre of $\omega sl_q(2)$ and give the analogous statements by applying the modification of Harish-Chandra homomorphism. Let $Y = \{E^iF^j(1-J) \mid i \geq 0,\ j \geq 0\}$,

$$C_q = EFJ + \frac{q^{-1}K + q\overline{K}}{(q-q^{-1})^2} = FEJ + \frac{qK + q^{-1}\overline{K}}{(q-q^{-1})^2},$$

and

$$P[K,\ \overline{K}] = \left\{ \sum_{i\geq 0} a_i K^i + \sum_{j>0} b_j \overline{K}^j \,\middle|\, J = K\overline{K} = \overline{K}K,\ K = KJ,\ J\overline{K} = J \right\},$$

where we set $K^0 = \overline{K}^0 = J$, $a_i, b_j \in \mathbb{C}$ $(i \geq 0, j > 0)$. Let Z_q be a polynomial algebra generated by the element C_q and J. It is shown that the centre of $\omega sl_q(2)$ is $Z_q \oplus Y$ and the restriction of the Harish-Chandra homomorphism to Z_q is an isomorphism onto the sub-algebra of $P[K,\ \overline{K}]$ generated by $qK + q^{-1}\overline{K}$.

This paper is organized as follows. Some basic facts and concepts are reviewed in Section 1. Then we attempt to get the isomorphism theorem for $\omega sl_q(2)$ in Section 2. Finally we devote to get the statements about the centre of $\omega sl_q(2)$ in the last section.

1. Preliminaries

There are at least two generalizations of a Hopf algebra, which are called weak Hopf algebras. One of them was introduced and studied in [7, 8]. In this sense the weak Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon)$ is just both bialgebra and there exists a so-called weak antipode $T \in Hom_k(H, H)$ of H such that $T * I * T = T$, $I * T * I = I$, where I is an identity map of H. Another definition of a weak Hopf algebra was introduced in [2]. The earlier proposals of face algebras [5], generalized Kac algebras [12] are weak Hopf algebras in this sense. However, the above two definitions of weak Hopf algebras are not included in each other.

One knows that the quantum algebra $\omega sl_q(2)$ is a weak Hopf algebra in the sense of [7]. The comultiplication Δ, the counit ε and the weak antipode T are given by the following formulas

$$\begin{aligned}
&\Delta(E) = 1 \otimes E + E \otimes K,\ \Delta(F) = F \otimes 1 + \overline{K} \otimes F,\\
&\Delta(K) = K \otimes K,\ \Delta(\overline{K}) = \overline{K} \otimes \overline{K},\\
&\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(\overline{K}) = 1,\\
&T(E) = -E\overline{K},\ T(F) = -KF,\ T(K) = \overline{K},\ T(\overline{K}) = K.
\end{aligned}$$

It is noticed that $J \neq 0$. If $J = 1$, $\omega sl_q(2)$ is isomorphic to $U_q(sl_2)$. Recall that the quantum algebra $U_q(sl_2)$, is generated by E' , F' , K' , K'^{-1} with the relations:

$$
\begin{aligned}
&K'^{-1}K' = K'K'^{-1} = 1,\\
&K'E'K'^{-1} = q^2E' \ , K'F'K'^{-1} = q^{-2}F',\\
&E'F' - F'E' = \frac{K' - K'^{-1}}{q - q^{-1}}.
\end{aligned}
$$

$U_q(sl_2)$ is of Hopf algebra structure, the comulitplication and antipode are

$$
\begin{aligned}
&\Delta(E') = 1 \otimes E' + E' \otimes K',\ \Delta(F') = F' \otimes 1 + K'^{-1} \otimes F',\\
&\Delta(K') = K' \otimes K',\ \Delta(K') = K' \otimes K',\\
&\varepsilon(E') = \varepsilon(F') = 0,\ \varepsilon(K') = \varepsilon(K'^{-1}) = 1,\\
&S(E') = -E'K'^{-1},\ S(F) = -K'^{-1}F,\ S(K') = K'^{-1},\ S(K'^{-1}) = K'.
\end{aligned}
$$

Accordingly, we always assume that $J \neq 0$ and $J \neq 1$.

Let $W = \omega sl_q(2)J$ and $Y = \omega sl_q(2)(1 - J)$.

Lemma 1.1. ([9, Theorem 4]) *As ideals of $\omega sl_q(2)$ we have $\omega sl_q(2) = W \oplus Y$. Moreover, $W \cong U_q(sl_2)$ as Hopf algebras. The basis of W is*

$$\{E^iF^jK^l,\ E^iF^j\overline{K}^m,\ E^iF^jJ \mid i \geq 0,\ j \geq 0,\ l > 0,\ m > 0\}$$

and the basis of Y is

$$\{E^iF^j(1 - J) \mid i \geq 0,\ j \geq 0\}.$$

Proof. We skecth the proof as follows.

It is easy to see that J is a central idempotent. Therefore, $\omega sl_q(2)J$ as well as $\omega sl_q(2)(1 - J)$ are ideals of $\omega sl_q(2)$. Hence,

$$\omega sl_q(2) = \omega sl_q(2)J \oplus \omega sl_q(2)(1 - J)$$

as ideals. One can see that W is of the basis

$$\{E^iF^jK^l,\ E^iF^j\overline{K}^m,\ E^iF^jJ \mid i \geq 0,\ j \geq 0,\ l > 0,\ m > 0\}$$

and Y has the basis $\{E^iF^j(1 - J) \mid i \geq 0,\ j \geq 0\}$. In fact, W is a Hopf algebra (the identity of W is J), the co-multiplication Δ is

$$
\begin{aligned}
&\Delta(EJ) = J \otimes EJ + EJ \otimes K,\\
&\Delta(FJ) = FJ \otimes J + \overline{K} \otimes FJ,\\
&\Delta(K) = K \otimes K,\ \Delta(\overline{K}) = \overline{K} \otimes \overline{K}.
\end{aligned}
$$

The counit ε is

$$\varepsilon(EJ) = \varepsilon(FJ) = 0,\ \varepsilon(K) = \varepsilon(\overline{K}) = 1$$

and the antipode is

$$T(EJ) = -E\overline{K},\ T(FJ) = -KF,\ T(K) = \overline{K},\ T(\overline{K}) = K.$$

Now let ρ be the algebra morphism from $U_q(sl_2)$ to W subjecting to

$$\rho(E') = EJ,\ \rho(F') = FJ,\ \rho(K') = K,\ \rho(K'^{-1}) = \overline{K}.$$

It is straightforward to see that ρ is a Hopf algebra isomorphism.

Let $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ for $m \geq 0$ and

$$[m]! = [1][2]\cdots[m],\ [0]! = 1,\ \begin{bmatrix} n \\ t \end{bmatrix} = \frac{[n]!}{[t]![n-t]!}.$$

We have

$$EF^m = F^mE + [m]F^{m-1}\frac{q^{-(m-1)}K - q^{m-1}\overline{K}}{q - q^{-1}}.$$

Let V be a $\omega sl_q(2)$-module and $0 \neq v \in V$. If $Kv = \lambda v$ and $\overline{K}v = \overline{\lambda}v$ for $\lambda, \overline{\lambda} \in \mathbb{C}$, we can conclude that if $\lambda \neq 0$, $\overline{\lambda} = \lambda^{-1}$ and if $\lambda = 0$, $\overline{\lambda} = 0$. We fix such a number $\overline{\lambda}$ which is corresponding to λ. We denote by V^λ the subspace of all vectors v in V such that $Kv = \lambda v$. The scalar λ is called a weight of V if $V^\lambda \neq 0$. An element $v \neq 0$ of V is said to be a highest weight vector of weight λ if $Ev = 0$ and $Kv = \lambda v$. A $\omega sl_q(2)$-module is said to be a highest weight module of highest weight λ if it is generated by a highest weight vector of λ.

Given a $\lambda \in \mathbb{C}$, we consider an infinite-dimensional vector space $V(\lambda)$ with basis $\{v_i\}_{i\in\mathbb{N}}$. For $p \geq 0$, we set

$$Kv_p = \lambda q^{-2p}v_p,\ \overline{K}v_p = \overline{\lambda}q^{2p}v_p, \tag{6}$$

$$Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\overline{\lambda}}{q - q^{-1}}v_{p-1}, \tag{7}$$

$$Fv_{p-1} = [p]v_p,\ Ev_0 = 0. \tag{8}$$

Lemma 1.2. *Relations* (6)-(8) *define a* $\omega sl_q(2)$*-modules structure on* $V(\lambda)$*. The element* v_0 *generates* $V(\lambda)$ *as a* $\omega sl_q(2)$*-module and is a highest weight vector of weight* λ *such that* $V(\lambda)$ *is the highest weight module.*

Proof. Let $l = \lambda\overline{\lambda}$. It is noticed that $\lambda \neq 0$ if and only if $\overline{\lambda} \neq 0$. Also if $\lambda \neq 0$ then $l = 1$ and if $\lambda = 0$, then $l = 0$. Therefore, either $\lambda \neq 0$ or $\lambda = 0$, we have $\lambda^2\overline{\lambda} = \lambda$ and

$\lambda\overline{\lambda}^2 = \overline{\lambda}$. Immediate computations yield

$$\begin{aligned}
&\overline{K}Kv_p = K\overline{K}v_p = lv_p,\ K\overline{K}Kv_p = Kv_p,\ \overline{K}K\overline{K}v_p = \overline{K}v_p,\\
&KEv_p = q^2EKv_p,\ \overline{K}Ev_p = q^{-2}E\overline{K}v_p,\\
&KFv_p = q^{-2}FKv_p,\ \overline{K}Fv_p = q^2F\overline{K}v_p,\\
[E,F]\,v_p &= ([p+1]\frac{q^{-p}\lambda - q^p\overline{\lambda}}{q-q^{-1}} - [p]\frac{q^{-(p-1)}\lambda - q^{p-1}\overline{\lambda}}{q-q^{-1}})v_p\\
&= \frac{q^{-2p}\lambda - q^{2p}\overline{\lambda}}{q-q^{-1}}v_p = \frac{K-\overline{K}}{q-q^{-1}}v_p.
\end{aligned}$$

This shows that the relations (6)-(8) define a $\omega sl_q(2)$-module structure on $V(\lambda)$. On the other hand, we have $Kv_0 = \lambda v_0$ and $Ev_0 = 0$, which means that v_0 is a highest weight vector of weight λ. Finally, (8) implies that $v_p = \frac{1}{[p]!}F^pv_0$ for all p, which proves that $V(\lambda)$ is generated by v_0.

The highest weight $\omega sl_q(2)$-module $V(\lambda)$ is called the Verma module of highest weight λ. We will apply the Verma module $V(\lambda)$ to give a description of the centre of $\omega sl_q(2)$.

2. Isomorphisms Among Weak Quantum Algebras

We now investigate the isomorphisms among weak quantum algebras.

Let $U_p(sl_2)$ be the algebra generated by E', F', K', K'^{-1} and the relations as that of $U_q(sl_2)$ where q is replaced by p. It is also a Hopf algebra with the same comultiplications as $U_q(sl_2)$.

The following lemma gives a condition that $U_p(sl_2) \cong U_q(sl_2)$ as Hopf algebras.

Lemma 2.1. *$U_p(sl_2) \cong U_q(sl_2)$ as Hopf algebras if and only if $p = \pm\, q^{\pm\,1}$.*

Proof. For convenience, we replace the generators E', F', K', K'^{-1} of $U_q(sl_2)$ by E, F, K, K^{-1}. The abuse notations are used in the proof.

Let $\phi : U_p(sl_2) \to U_q(sl_2)$ be a bialgebra isomorphism, then we have

$$\Delta(\phi(E')) = (\phi\otimes\phi)(\Delta(E')) = 1\otimes\phi(E') + \phi(E')\otimes\phi(K'). \tag{9}$$

Note that $\phi(K')$ is necessarily a group-like element. Therefore, $\phi(E')$ is a skew-primitive element in $U_q(sl_2)$.

By Theorem 5.4.1, Lemma 5.5.5, the subsequent comments in [10], and [4, Theorem A], we can assume that

$$\phi(K') = K,\ \phi(E') = aE + bFK + c(1-K).$$

Then (9) automatically holds. Applying ϕ to the equation $K'E' = p^2E'K'$, we yield that

$$K(aE + bFK + c(1-K)) = p^2(aE + bFK + c(1-K))K.$$

Consequently, we get

$$0 = a(q^2 - p^2) = b(p^2 - q^{-2}) = c(1 - p^2).$$

It follows that $c = 0$ since p is not a root of unity.

If $a \neq 0$, then $p^2 = q^2$ and $b = 0$. In this case, we get that $\phi(E) = aE$ and $p = \pm q$. If this is the case, we get that $\phi(F) = \pm a^{-1}F,\ K \to K,\ K^{-1} \to K^{-1}$.

If $b \neq 0$, then $p^2 = q^{-2}$ and $a = 0$. In this case, we get that $\phi(E) = bFK$ and $p = \pm q^{-1}$. Similarly, $\phi(F) = \pm\, b^{-1}K^{-1}E$.

Conversely, if $p = \pm q$, it is obvious that $\psi : U_p(sl_2) \cong U_q(sl_2)$ defined by

$$\psi(E') = E,\ \psi(F') = \pm F,\ \psi(K') = K,\ \psi(K'^{-1}) = K^{-1}$$

is a Hopf algebra isomorphism.

If $p = \pm q^{-1}$, then $\psi : U_p(sl_2) \cong U_q(sl_2)$ defined by

$$\psi(E') = FK,\ \psi(F') = \pm K^{-1}E,\ \psi(K') = K,\ \psi(K'^{-1}) = K^{-1}$$

is a Hopf algebra isomorphism.

Let $\omega sl_p(2)$ be the algebra generated by E, F, K, K^{-1} and the relations as that of $\omega sl_q(2)$ where q is replaced by p. It is a weak algebra with the same comultiplications as $\omega sl_q(2)$.

Lemma 2.2. *Let $x \in Y$ and $b \neq 0$. If $\Delta(x) = b(1-J)\otimes EJ + 1\otimes x + x\otimes K$, then $x = bE(1-J)$.*

Proof. Let $x = \sum_{s,t}\xi(s,t)E^sF^t(1-J)$. By the assumption, we have

$$\Delta(x) = \sum_{s,t}\xi(s,t)E^sF^t(1-J)\otimes K + \sum_{s,t}\xi(s,t)1\otimes E^sF^t(1-J) + b(1-J)\otimes EJ.$$

On the other hand, if $j > 0$, then $K^jJ = K^j$ and $\overline{K}^jJ = \overline{K}^j$. One easily sees that

$$\begin{aligned}
\Delta(x) = &\left(\sum_{s,t,i,j}\xi(s,t)q^{i(s-i)}q^{j(t-j)}\begin{bmatrix} s \\ i\end{bmatrix}\begin{bmatrix} t \\ j\end{bmatrix}E^iF^{t-j}\overline{K}^j\otimes E^{s-i}K^iF^j\right)\\
&-\left(\sum_{s,t,i,j}\xi(s,t)q^{i(s-i)}q^{j(t-j)}\begin{bmatrix} s \\ i\end{bmatrix}\begin{bmatrix} t \\ j\end{bmatrix}E^iF^{t-j}\overline{K}^jJ\otimes E^{s-i}K^iF^jJ\right)\\
= &\left(\sum_{s,t,j\neq 0}\xi(s,t)q^{j(t-j)}\begin{bmatrix} t \\ j\end{bmatrix}F^{t-j}\overline{K}^j\otimes E^sF^j(1-J)\right)\\
&+\left(\sum_{s,t,i\neq 0}\xi(s,t)q^{i(s-i)}\begin{bmatrix} s \\ i\end{bmatrix}E^iF^t(1-J)\otimes E^{s-i}K^i\right)\\
&+\left(\sum_{s,t}\xi(s,t)F^t\otimes E^s\right)-\left(\sum_{s,t}\xi(s,t)F^tJ\otimes E^sJ\right)
\end{aligned}$$

Comparing the above two equality for $\Delta(x)$, all $t = 0$ and $s = 1$. Hence we can assume that $x = aE(1 - J)$ and we get

$$\Delta(x) = a(1 - J) \otimes E(1 - J) + 1 \otimes x + x \otimes K.$$

It follows that $a = b$ and $x = bE(1 - J)$.

The same argument shows that there is no element $x \in Y$ such that $\Delta(x) = x \otimes K + 1 \otimes x + b(1 - J) \otimes FK$ and $x \in Y$ where $b \neq 0$.

The main result of this section is as follows.

Theorem 2.3. *$\omega sl_q(2) \cong \omega sl_p(2)$ as weak Hopf algebras if and only if $p = \pm\, q$.*

Proof. Let $\gamma : \omega sl_q(2) \cong \omega sl_p(2)$ be a weak Hopf algebra isomorphism. One knows that γ sends group-likes to group-likes, now it is easy to see that $\gamma(J) = J$.

According to Lemma 1.1, $\omega sl_q(2) = W \oplus Y$, $W \cong U_q(sl_2)$ as Hopf algebras; $\omega sl_p(2) = W' \oplus Y'$, $W' \cong U_p(sl_2)$ as Hopf algebras, where Y, Y' are spanned respectively by the same set $\{E^i F^j(1 - J) \mid i \geq 0,\ j \geq 0\}$ as an ideal of $\omega sl_q(2)$ and $\omega sl_p(2)$.

Let $\mathtt{inj}_q : W \to \omega sl_q(2)$ be the inclusion defined by

$$J \to J,\ EJ \to EJ,\ FJ \to FJ,\ K \to K,\ \overline{K} \to \overline{K},$$

and then extend it by linearity. It is easy to see that $\mathtt{inj}_q$ is a weak Hopf algebra injection. Indeed, $\mathtt{inj}_q$ is an algebra homomorphism. For the relation (3),

$$\mathtt{inj}_q(K)\mathtt{inj}_q(EJ) = KEJ = q^2 EJK = q^2 \mathtt{inj}_q(EJ)\mathtt{inj}_q(K).$$

The rest of (3) and the relations (4) are similar. For the relation (5),

$$\mathtt{inj}_q(EJ)\mathtt{inj}_q(FJ) - \mathtt{inj}_q(FJ)\mathtt{inj}_q(EJ) = (EF - FE)J = \frac{\mathtt{inj}_q(K) - \mathtt{inj}_q(\overline{K})}{q - q^{-1}}.$$

The map $\mathtt{inj}_q$ is also a coalgebra map. Indeed,

$$\Delta(\mathtt{inj}_q(EJ)) = \Delta(EJ) = J \otimes EJ + EJ \otimes K$$

and

$$(\mathtt{inj}_q \otimes \mathtt{inj}_q)\Delta(EJ) = (\mathtt{inj}_q \otimes \mathtt{inj}_q)(J \otimes EJ + EJ \otimes K) = J \otimes EJ + EJ \otimes K.$$

$$\Delta(\mathtt{inj}_q(EJ)) = (\mathtt{inj}_q \otimes \mathtt{inj}_q)\Delta(EJ).$$

Similarly, we have $\Delta(\mathtt{inj}_q(X)) = (\mathtt{inj}_q \otimes \mathtt{inj}_q)\Delta(X)$ where $X = FJ,\ K,\ \overline{K}$ or J. It is easy to see that $W' = \mathrm{im}(\gamma \circ \mathtt{inj}_q)$ since $\gamma(J) = J$. This implies that if $\gamma : \omega sl_q(2) \to \omega sl_p(2)$ is a weak Hopf algebra isomorphism, then $U_p(sl_2) \cong U_q(sl_2)$ as Hopf algebras. By Lemma 2.1, $p = \pm\, q^{\pm\, 1}$. However, if $p = \pm q^{-1}$, we must have

$$\gamma(EJ) = b(FJ)K,\ \gamma(FJ) = \pm b^{-1}\overline{K}(EJ), \gamma(K) = K, \gamma(\overline{K}) = \overline{K}$$

for some $b \neq 0$. If there is a way to extend it to $\omega sl_q(2)$ such that γ is a weak Hopf algebra isomorphism, we assume that $\gamma(E(1-J)) = x$, then $0 \neq x \in Y$ and $\gamma(E) = \gamma(EJ + E(1-J)) = b(FJ)K + x$. Since γ is a weak Hopf algebra isomorphism, we have

$$\Delta(b(FJK) + x) = (b(FJ)K + x) \otimes K + 1 \otimes (b(FJ)K + x).$$

Hence, $\Delta(x) = x \otimes K + 1 \otimes x + b(1-J) \otimes FK$. It is impossible, so $p = \pm q$.

Conversely, if $p = \pm q$, we set

$$\gamma(E) = E, \gamma(F) = \pm\, F,\ \gamma(K) = K,\ \gamma(\overline{K}) = \overline{K}.$$

It is easy to see that γ is a weak Hopf algebra isomorphism.

The proof is completed.

Now we can determine all such isomorphisms. Indeed, if $\gamma : \omega sl_q(2) \cong \omega sl_p(2)$ is a isomorphism of weak Hopf algebra, then $p = \pm\, q$. Furthermore, $\gamma \circ \mathtt{inj}_q$ is an isomorphism of Hopf algebras between W and W', defined by

$$J \to J, EJ \to aEJ, FJ \to \pm a^{-1}FJ, K \to K,\ \overline{K} \to \overline{K}$$

by Lemma 2.1. The map γ restricted to W must be of this form. To get the map γ, we assume that $\gamma(E(1-J)) = x$, it is easy to see that $\gamma(E) = aEJ + x$ and $x \in Y$. Since γ is a weak Hopf algebra isomorphism, we then get that $\Delta(x) = 1 \otimes x + x \otimes K + a(1-J) \otimes EJ$. By Lemma 2.2, we have $x = aE(1-J)$. Similarly, we also have $\gamma(F(1-J)) = \pm a^{-1}F(1-J)$. This implies that γ has to be $J \to J, E \to aE, F \to \pm a^{-1}F, K \to K,\ \overline{K} \to \overline{K}$ and extended linearity.

3. The Centre of $\omega sl_q(2)$

In [13], the authors introduce a new quantum algebra $U_q(f(H,K))$, which generalizes the quantum group $U_q(sl_2)$. Then they obtained statements about its centre by applying the Harish-Chandra homomorphism. In this section, we give the similar description about the centre of $\omega sl_q(2)$. Recall that

$$P[K,\ \overline{K}] = \left\{ a_0 J + \sum_{i>0} a_i K^i + \sum_{j>0} b_j \overline{K}^j \,\middle|\, J = K\overline{K} = \overline{K}K,\ K = KJ,\ J\overline{K} = J \right\}.$$

We set $K^0 = J = \overline{K}^0$ for convenience.

Keeping all notations as the previous sections. Let Z_q denote the centre of W and Z_ω the centre of $\omega sl_q(2)$. To state our main result, several lemmas are needed as follows.

Lemma 3.1. $Y \subseteq Z_\omega$.

Proof. It is noticed that

$$Y = \{E^i F^j (1-J) \mid i \geq 0, j \geq 0\}.$$

Since

$$\begin{aligned}E(E^iF^j(1-J)) &= E^i(F^jE+[j]F^{j-1}\frac{q^{-(j-1)}K-q^{j-1}\overline{K}}{q-q^{-1}})(1-J)\\&= E^iF^j(1-J)E,\\F(E^iF^j(1-J)) &= (E^iF-[i]E^{i-1}\frac{q^{i-1}K-q^{-(i-1)}\overline{K}}{q-q^{-1}})F^j(1-J)\\&= E^iF^j(1-J)F,\\K(E^iF^j(1-J)) &= q^{2i-2j}E^iF^j(1-J)K=0=E^iF^j(1-J)K,\\\overline{K}(E^iF^j(1-J)) &= q^{2i-2j}E^iF^j(1-J)\overline{K}=0=E^iF^j(1-J)\overline{K}.\end{aligned}$$

The result follows.

Let

$$C_q=EFJ+\frac{q^{-1}K+q\overline{K}}{(q-q^{-1})^2}=FEJ+\frac{qK+q^{-1}\overline{K}}{(q-q^{-1})^2}. \tag{10}$$

It is called the J-quantum Casimir element.

Let W^K be the sub-algebra of W consisting of all elements commuting with K. For any $x\in W^K$, then $xK=Kx$ and $xJ=Jx=x$. It follows that

$$\overline{K}Jx=\overline{K}xJ=Jx\overline{K}=xJ\overline{K}.$$

Hence $\overline{K}x=x\overline{K}$ and the elements of W^K commute with $\overline{K}$.

Let $I=WE\cap W^K$, it is a left ideal of W^K.

The following three lemmas are very similar to [6, Lemma VI.4.2-Lemma VI. 4.3] and their proofs are more or less the same.

Lemma 3.2. *The element $C_q\in Z_\omega$.*

Lemma 3.3. *Any element of W belongs to W^K if and only if it is of the form $\sum_{i\geq 0}F^iP_iE^i$, where $P_0,\ P_1,\ \cdots$ are elements of $P[K,\ \overline{K}]$.*

Lemma 3.4. *We have $I=FW\cap W^K$ and $W^k=P[K,\ \overline{K}]\oplus I$.*

It results from $I=FW\cap W^K$ that I is a two-sided ideal and that the projection φ from W^K onto $P[K,\ \overline{K}]$ is a morphism of algebras. The map φ is called the Harish-Chandra homomorphism. It permits one to express the action of the centre Z_q of W on a highest weight module.

The following lemmas are similar to [6, Lemma VI.4.4-Lemma VI. 4.7], but details in the proofs have to be changed to suit for our cases. For completeness, we write them down here.

Lemma 3.5. *Let V be a highest weight $\omega sl_q(2)$-module with highest weight λ. Then, for any central element z of W and any $v\in V$, we have $zv=\varphi(z)(\lambda,\ \overline{\lambda})v$, where $\varphi(z)$ is element of $P[K,\ \overline{K}]$ and that $\varphi(z)(\lambda,\ \overline{\lambda})$ is its value at $\lambda,\ \overline{\lambda}$.*

Proof. Let v_0 be a highest weight vector generating V and z is a central element of W, the element z can be written in the form

$$z = \varphi(z) + \sum_{i>0} F^i P_i E^i.$$

Since $Ev_0 = 0$ and $Kv_0 = \lambda v_0$, $\overline{K}v_0 = \overline{\lambda}v_0$, we get $zv_0 = \varphi(z)(\lambda,\ \overline{\lambda})v_0$. If v is an arbitrary element of V, we have $v = xv_0$ for some $x \in \omega sl_q(2)$. It is noticed that $x = x_1 + x_2$ where $x_1 \in W$ and $x_2 \in Y$. Since $YW = WY = 0$, $zx_2 = x_2 z = 0$ and $zx = xz$. Hence

$$zv = zxv_0 = xzv_0 = \varphi(z)(\lambda,\ \overline{\lambda})xv_0 = \varphi(z)(\lambda,\ \overline{\lambda})v.$$

The result follows.

We now consider the restriction of the Harish-Chandra homomorphism to Z_q.

Lemma 3.6. *Let $z \in Z_q$ and if $\varphi(z) = 0$, then $z = 0$.*

Proof. Let z be an element in the centre of W such that $\varphi(z) = 0$. Assume z non-zero, it can be written as $z = \sum_{i=k}^{l} F^i P_i E^i$ where $0 < k \leq l$ are integers and $P_k, \cdots, P_l$ are non-zero elements of $P[K,\ \overline{K}]$. Consider the Verma module $V(\lambda)$ whose highest weight is neither a power of q or 0 (therefore, $\overline{\lambda} = \lambda^{-1}$). The relations (6)-(8) show that $Ev_p = 0$ if and only if $p = 0$. We apply z to the vector v_k of $V(\lambda)$, on one hand, Lemma 3.6 implies that $zv_k = \varphi(z)(\lambda,\ \overline{\lambda}) = 0$, on the other hand, we get $zv_k = F^k P_k E^k v_k = cP_k(\lambda,\ \overline{\lambda})v_k$ where c is a non-zero constant. It follows that $P_k(\lambda,\ \overline{\lambda}) = 0$. As a consequence, we have a non-zero polynomial with infinitely many roots. It is a contradiction.

For any element Q of $P[K, \overline{K}]$, denoted by $\widetilde{Q}$ the polynomial defined by the change of variable $\widetilde{Q}(\lambda, \overline{\lambda}) = Q(q^{-1}\lambda,\ q\overline{\lambda})$.

Lemma 3.7. *For any element z in Z_q, we have $\widetilde{\varphi}(z)(\lambda,\ \overline{\lambda}) = \widetilde{\varphi}(z)(\overline{\lambda},\ \lambda)$.*

Proof. If $\lambda = 0$, then $\overline{\lambda} = 0$, the result is obvious. The following is under the assumption that $\lambda \neq 0$. Therefore, $\overline{\lambda} = \lambda^{-1}$. For any integer $n > 0$ consider the Verma module $V(q^{n-1})$. By the formula (7), we have

$$Ev_n = \frac{q^{-(n-1)}q^{n-1} - q^{n-1}q^{-(n-1)}}{q - q^{-1}} v_{n-1} = 0.$$

Thus, v_n is a highest weight vector of weight $q^{n-1-2n} = q^{-n-1}$. By Lemma 3.5, a central element z acts on the module generated by v_n as the multiplication by the scalar $\varphi(z)(q^{-n-1},\ q^{n+1})$, but since v_n is in $V(q^{n-1})$, then

$$zv_n = \varphi(z)(q^{n-1},\ q^{-n+1})v_n.$$

In other words, we have

$$\widetilde{\varphi}(z)(q^n,\ q^{-n}) = \widetilde{\varphi}(z)(q^{-n},\ q^n).$$

The lemma follows.

Lemma 3.8. *Any polynomial of* $P[K, \overline{K}]$ *satisfying the relation* $Q(\lambda, \overline{\lambda}) = Q(\overline{\lambda}, \lambda)$ *is a polynomial in* $k[K + \overline{K}]$.

Proof. We proceed by induction on the degree of the polynomial on K. If the degree is 0, the statement holds trivially. Suppose that the lemma is proved for all degrees $< n$ and let Q be element of degree n for K such that $Q(\lambda,\ \overline{\lambda}) = Q(\overline{\lambda},\ \lambda)$.
Then we may write Q in the form

$$Q = c(K^n + \overline{K}^n) + \text{ (terms of degree } < n).$$

Now

$$K^n + \overline{K}^n = (K + \overline{K})^n + (\text{terms of degree} < n),$$

where we set $(K + \overline{K})^0 = J$, $J^2 = J = K\overline{K}$. One concludes by applying the induction hypothesis.

We are ready to prove our main theorem.

Theorem 3.9. *When* q *is not a root of unity, the centre of* $\omega sl_q(2)$ *is* $Z_q \oplus Y$, *where* Z_q *is a polynomial algebra generated by the element* C_q *and* J. *The restriction of the Harish-Chandra homomorphism to* Z_q *is an isomorphism onto the sub-algebra of* $P[K,\ \overline{K}]$ *generated by* $qK + q^{-1}\overline{K}$.

Proof. We have already known that the restriction of φ to the Z_q is injective by Lemma 3.6. We are left to determine its image. By Lemma 3.7 and Lemma 3.8, the latter is contained in the sub-algebra of $P[K,\ \overline{K}]$ generated by $qK + q^{-1}\overline{K}$. Consider the central element C_q defined by (10), we know that

$$\varphi(C_q) = \frac{1}{(q - q^{-1})^2}(qK + q^{-1}\overline{K}),\ \ \varphi(K\overline{K}) = K\overline{K},$$

which proves that the image of Z_q is the whole sub-algebra and that C_q and J generate Z_q. The latter is a polynomial algebra generated by C_q and J. By Lemma 1.1, $\omega sl_q(2) = W \oplus Y$. It follows that $Z_\omega = Z_q \oplus Y$.

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