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A NOT ON $|\overline{N}, p_n; \delta|_k$ SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. In this paper a general theorem on $|\overline{N}, p_n; \delta|_k$ summability factors which generalizes a theorem of Bor [4] on for $|\overline{N}, p_n|_k$ summability factors of infinite series.

1. Definition and notion

Let $\sum a_n$ be a given infinte series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (u_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficient (p_n) [6].

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$ if [1]

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |u_n - u_{n-1}|^k < \infty,$$

and it is said to be summable $|\overline{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if [2]

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |u_n - u_{n-1}|^k < \infty, \tag{1.1}$$

In the special case when $\delta = 0$ (resp. $p_n = 1$ for all values of n) $|\overline{N}, p_n; \delta|_k$ summability is the same as $|\overline{N}, p_n|_k$ (resp. $|C, 1; \delta|_k$) summability.

Leindler [7] has generalized the notion of $|C, \alpha; \delta|_k$ summability replacing the function n^δ by a positive non-decreasing function $\delta(n)$. $(1 < n < \infty)$.

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Leindler [7] A series $\sum a_n$ is summable $|C,\alpha;\delta(n)|_k$ if the series

$$\sum_{n=1}^{\infty} \delta(n)^k n^{(k-1)} |\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}|^k$$

converges, where σ_n^{α} is the n^{th} Cesáro mean of order α of $\sum a_n$.

With $\delta(n) = n^{\delta}$, it follows that above definition reduces to that of Flett [5].

The equation (1.1) will be

$$\sum_{n=0}^{\infty} \delta\left(\frac{P_n}{p_n}\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.2}$$

2. Quite recently Bor [3] proved the following theorem for $|\overline{N}, p_n|_k$ summability factors of infinite series.

Theorem A. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \ \text{ as } \ n \to \infty. \tag{2.1}$$

Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (λ_n) and (β_n) such that

$$|\Delta\lambda_n| \le \beta_n; \tag{2.2}$$

$$\beta_n \to 0 \quad \text{as } \ n \to \infty; \tag{2.3}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta|X_n < \infty; \tag{2.4}$$

$$|\lambda_n|X_n = O(1) \ \text{ as } \ n \to \infty. \tag{2.5}$$

If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \ \text{ as } \ m \to \infty \tag{2.6}$$

where

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v, \tag{2.7}$$

then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \ge 1$.

The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n;\delta|_k$ summability. Now, we shall prove the following theorem.

3. Theorem

Let (X_n) be a positive non-decreasing sequence and the sequence (λ_n) and (β_n) be such that condition (2.2)–(2.5) of Theorem A are satisfied. If (p_n) is a sequence such that condition (1.1) of Theorem A is satisfied and

$$\sum_{n=v+1}^{\infty}\delta\Big(\frac{P_n}{p_n}\Big)^k\Big(\frac{p_n}{P_n}\Big)\frac{1}{P_{n-1}} = O\Big\{\delta\Big(\frac{P_v}{p_v}\Big)^k\frac{1}{P_v}\Big\}, \tag{3.1}$$

$$\sum_{n=1}^{m}\delta\Big(\frac{P_n}{p_n}\Big)^k\Big(\frac{p_n}{P_n}\Big)|t_n|^k = O(X_m) \text{ as } m\to\infty, \tag{3.2}$$

where (t_n) is as in (2.7), then the series $\sum a_n\lambda_n$ is summable $|\overline{N},p_n;\delta|_k$ for $k\geq 1$, and $0\leq\delta<\frac{1}{k}$.

Remark. It may be noted that, if we take $\delta=0$ in this theorem, then we get Theorem A. In this case condition (3.2) reduces to condition (2.6) and condition (3.1) reduces to

$$\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}} = O\Big(\frac{1}{P_v}\Big),$$

which always holds.

We need the following lemma for the proof of our theorem.

Lemma. ([6]) If the condition (2.2)–(2.5) on (X_n), (β_n) and (λ_n) are satisfied, then

$$n\beta_nX_n = O(1) \text{ as } m\to\infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty}\beta_nX_n<\infty. \tag{3.4}$$

4. Proof of the theorem

Let (T_n) be the sequence of $(\overline{N},p_n)$ means of the series $\sum a_n\lambda_n$. Then by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{i=0}^{v}a_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v.$$

Then, for $n\geq 1$, we have

$$T_n-T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v.$$

Using Abel's transformation, we get

$$\begin{aligned} T_n-T_{n-1} &= \frac{n+1}{nP_n}p_nt_n\lambda_n - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v} \\ &\quad+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\Delta\lambda_vt_v\frac{v+1}{v}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vt_v\lambda_{v+1}\frac{1}{v}\end{aligned}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}$$

Since

$$|T^{\alpha}_{n,1} + T^{\alpha}_{n,2} + T^{\alpha}_{n,3} + T^{\alpha}_{n,4}|^k \le 4^k\left(|T^{\alpha}_{n,1}|^k + |T^{\alpha}_{n,2}|^k + |T^{\alpha}_{n,3}|^k + |T^{\alpha}_{n,4}|^k\right)$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{P_n}{p_n}\Big)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1,2,3,4.$$

Since

$$\lambda_n = O\Big(\frac{1}{X_n}\Big) = O(1), \text{ by (2.5) we get that}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{P_n}{p_n}\Big)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{p_n}{P_n}\Big) |t_n|^k \\
&= O(1) \sum_{n=1}^{m} |\lambda_n| \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{p_n}{P_n}\Big) |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \delta\Big(\frac{P_v}{p_v}\Big)^k \Big(\frac{p_v}{P_v}\Big) |t_v|^k \\
&\quad + O(1)|\lambda_m| \sum_{n=1}^{m} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{p_n}{P_n}\Big) |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m \\
&= O(1) \text{ as } m \to \infty, \text{ by (2.2), (3.2) and (3.4).}
\end{aligned}$$

Now applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{P_n}{p_n}\Big)^{k-1} |T_{n,2}|^k &= O(1) \sum_{n=2}^{m+1} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{p_n}{P_n}\Big) \frac{1}{P_{n-1}} \\
&\quad \times \Big\{\sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k\Big\} \Big\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\Big\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} |\lambda_v| |t_v|^k \sum_{n=v+1}^{m+1} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{p_n}{P_n}\Big) \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} |\lambda_v| \delta\Big(\frac{P_v}{p_v}\Big)^k \Big(\frac{p_v}{P_v}\Big) |t_v|^k \\
&= O(1) \text{ as } m \to \infty,
\end{aligned}$$

using the fact that $P_v = O(v p_v)$, by (2.1) and $n\beta_n = O(\frac{1}{X_n}) = O(1)$, by (3.3), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{P_n}{p_n}\Big)^{k-1} |T_{n,3}|^k &= O(1) \sum_{n=2}^{m+1} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{p_n}{P_n}\Big) \frac{1}{P_{n-1}} \\
&\quad \times \Big\{ \sum_{v=1}^{n-1} (v\beta_v)^k p_v |t_v|^k \Big\} \Big\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \Big\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} (v\beta_v)(v\beta_v)^{k-1} p_v |t_v|^k \sum_{n=v+1}^{\infty} \delta\Big(\frac{P_n}{p_n}\Big)^k \times \Big(\frac{p_n}{P_n P_{n-1}}\Big) \\
&= O(1) \sum_{v=1}^{m} (v\beta_v) \delta\Big(\frac{P_v}{p_v}\Big)^k \Big(\frac{p_v}{P_v}\Big) |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{i=1}^{v} \delta\Big(\frac{P_i}{p_i}\Big)^k \Big(\frac{p_i}{P_i}\Big) |t_i|^k \\
&\quad + O(1) m\beta_m \sum_{v=1}^{m} \delta\Big(\frac{P_v}{p_v}\Big)^k \Big(\frac{p_v}{P_v}\Big) |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v X_v |\Delta\beta_v| + O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_v + O(1) m\beta_m X_m \\
&= O(1) \ \text{ as } \ m \to \infty,
\end{aligned}$$

by (2.2), (2.4), (3.1), (3.2), (3.3) and (3.4).

Finally, using the fact that $P_v = O(v p_v)$, by (2.1), as in $T_{n,1}$ and $T_{n,2}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{P_n}{p_n}\Big)^{k-1} |T_{n,4}|^k &= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \delta\Big(\frac{P_v}{p_v}\Big)^k \Big(\frac{p_v}{P_v}\Big) |t_v|^k \\
&= O(1) \ \text{ as } \ m \to \infty.
\end{aligned}$$

Therefore, we get that

$$\sum_{n=1}^{m} \delta\Big(\frac{P_n}{p_n}\Big)^k \Big(\frac{P_n}{p_n}\Big)^{k-1} |T_{n,r}|^k = O(1) \ \text{ as } \ m \to \infty, \quad \text{for } \ r = 1, 2, 3, 4.$$

This completes the proof of the theorem.

If we take $p_n = 1$ for all values of n in this theorem, then we get a result concerning the $|C, 1; \delta|_k$ summability methods.

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