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GROWTH ESTIMATES ON CERTAIN INTEGRAL INEQUALITIES IN TWO VARIABLES INVOLVING ITERATED INTEGRALS

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Abstract. The aim of the present paper is to establish growth estimates on some new integral inequalities in two independent variables involving iterated double integrals, which can be used to study the qualitative behavior of solutions of certain partial integrodifferential and integral equations. Applications are also given to illustrate the usefulness of one of our results.

1. Introduction

Inequalities which provide explicit growth estimates on the unknown functions have played a fundamental role in the theory of differential, integral and finite difference equations. A detailed account on such inequalities and some of their applictions can be found in [1-4, 7, 8] and the references given therein. In [2] Bykov and Salpagarov (see also [1, 3, 4]) have established the explicit upper bounds on the following integral inequalities involving iterated integrals

$$u(t) \leq c + \int_\alpha^t k(t,s)u(s)ds + \int_\alpha^t \left(\int_\alpha^s h(t,s,\sigma)u(\sigma)d\sigma\right) ds,$$

$$u(t) \leq c + \int_\alpha^t b(s)u(s)ds + \int_\alpha^t \left(\int_\alpha^s k(s,\tau)u(\tau)d\tau\right) ds + \int_\alpha^t \left(\int_\alpha^s \left(\int_\alpha^\tau h(s,\tau,\sigma)u(\sigma)d\sigma\right) d\tau\right) ds,$$

under some suitable conditions on the functions involved therein. In the qualitative analysis of some general classes of partial integrodifferential and integral equations the bounds provided by the earlier inequalities in the literature are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. Motiviated by the above inequalities, in this paper we establish some new integral inequalities involving functions of two independent variables, which can be used as ready tools in the analysis

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of certain classes of partial integrodifferential and integral equations. Some applications of one of our results are also given.

2. Statement of Results

In what follows, R denotes the set of real numbers and $R_+ = [0,\infty)$, $I = [0,\alpha]$, $J = [0,\beta]$ are given subsets of R. We denote by $\Delta = I \times J$, $D = \{(x,y,s,t) \in \Delta^2 : 0 \leq s \leq x \leq \alpha, 0 \leq t \leq y \leq \beta\}$, $E = \{(x,y,s,t,\sigma,\tau) \in \Delta^3 : 0 \leq \sigma \leq s \leq x \leq \alpha, 0 \leq \tau \leq t \leq y \leq \beta\}$ and $C(A,B)$ the class of continuous functions from the set A to the set B. The partial derivatives of a function $z(x,y)$ x, $y \in R$ with respect to x, y and xy are denoted by $z_x(x,y)$, $z_y(x,y)$ and $z_{xy}(x,y)$ respectively.

Our main results are established in the following theorems.

Theorem 1. *Let* $u \in C(\Delta, R_+)$, $k(x,y,s,t)$, $k_x(x,y,s,t)$, $k_y(x,y,s,t)$, $k_{xy}(x,y,s,t) \in C(D,R_+)$, $h(x,y,s,t,\sigma,\tau)$, $h_x(x,y,s,t,\sigma,\tau)$, $h_y(x,y,s,t,\sigma,\tau)$, $h_{xy}(x,y,s,t,\sigma,\tau) \in C(E,R_+)$ *and* $c \geq 0$ *be a constant.*

(a_1) *If*

$$u(x,y) \leq c + \int_0^x \int_0^y k(x,y,s,t)u(s,t)dtds + \int_0^x \int_0^y \left(\int_0^s \int_0^t h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)d\tau d\sigma\right) dtds, \tag{2.1}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq c \exp\left(\int_0^x \int_0^y [A(m,n) + B(m,n)]dndm\right), \tag{2.2}$$

for $(x,y) \in \Delta$, *where*

$$A(x,y) = k(x,y,x,y) + \int_0^x k_x(x,y,\xi,y)d\xi + \int_0^y k_y(x,y,x,\eta)d\eta + \int_0^x \int_0^y k_{xy}(x,y,\xi,\eta)d\eta d\xi, \tag{2.3}$$

$$B(x,y) = \int_0^x \int_0^y h(x,y,x,y,\sigma,\tau)d\tau d\sigma + \int_0^x \left(\int_0^s \int_0^y h_x(x,y,s,y,\sigma,\tau)d\tau d\sigma\right) ds + \int_0^y \left(\int_0^x \int_0^t h_y(x,y,x,t,\sigma,\tau)d\tau d\sigma\right) dt + \int_0^x \int_0^y \left(\int_0^s \int_0^t h_{xy}(x,y,s,t,\sigma,\tau)d\tau d\sigma\right) dtds. \tag{2.4}$$

(a_2) *Let* $g(u)$ *be continuously differentialbe function defined for* $u \geq 0$, $g(u) > 0$ *for* $u > 0$ *and* $g'(u) \geq 0$ *for* $u \geq 0$. *If*

$$u(x,y) \leq c + \int_0^x \int_0^y k(x,y,s,t)g(u(s,t))dtds$$

$$+\int_0^x\int_0^y\left(\int_0^s\int_0^t h(x,y,s,t,\sigma,\tau)g(u(\sigma,\tau))d\tau d\sigma\right)dtds \tag{2.5}$$

for $(x,y)\in\Delta$, *then for* $0\le x\le x_1$, $0\le y\le y_1$; $x,\ x_1\in I$, $y,\ y_1\in J$,

$$u(x,y)\le G^{-1}\left[G(c)+\int_0^x\int_0^y[A(m,n)+B(m,n)]dndm\right], \tag{2.6}$$

where $A(x,y)$, $B(x,y)$ *are given by* (2.3), (2.4),

$$G(r)=\int_{r_0}^r\frac{dw}{g(w)},\quad r>0, \tag{2.7}$$

$r_0>0$ *is arbitrary,* G^{-1} *is the inverse of* G *and* $x_1\in I$, $y_1\in J$ *are chosen so that*

$$G(c)+\int_0^x\int_0^y[A(m,n)+B(m,n)]dndm\in Dom(G^{-1}),$$

for all $(x,y)\in\Delta$ *such that* $0\le x\le x_1$, $0\le y\le y_1$.

Theorem 2. *Let* $u(x,y)$, $a(x,y)\in C(\Delta,R_+)$, $k(x,y,s,t)\in C(D,R_+)$, $h(x,y,s,t,\sigma,\tau)\in C(E,R_+)$ *and* $c\ge 0$ *be a constant.*

(b_1) *If*

$$u(x,y)\le c+\int_0^x\int_0^y a(s,t)u(s,t)dtds+\int_0^x\int_0^y\left(\int_0^s\int_0^t k(s,t,\sigma,\tau)u(\sigma,\tau)d\tau d\sigma\right)dtds$$
$$+\int_0^x\int_0^y\left(\int_0^s\int_0^t\left(\int_0^\sigma\int_0^\tau h(s,t,\sigma,\tau,m,n)u(m,n)dndm\right)d\tau d\sigma\right)dtds, \tag{2.8}$$

for $(x,y)\in\Delta$, *then*

$$u(x,y)\le c\exp\left(\int_0^x\int_0^y Q(s,t)dtds\right), \tag{2.9}$$

for $(x,y)\in\Delta$, *where*

$$Q(x,y)=a(x,y)+\int_0^x\int_0^y k(x,y,\sigma,\tau)d\tau d\sigma$$
$$+\int_0^x\int_0^y\left(\int_0^\sigma\int_0^\tau h(x,y,\sigma,\tau,m,n)dndm\right)d\tau d\sigma. \tag{2.10}$$

(b_2) *Let* $g(u)$ *be as in Theorem* 1, *part* (a_2). *If*

$$u(x,y)\le c+\int_0^x\int_0^y a(s,t)g(u(s,t))dtds$$
$$+\int_0^x\int_0^y\left(\int_0^s\int_0^t k(s,t,\sigma,\tau)g(u(\sigma,\tau))d\tau d\sigma\right)dtds$$
$$+\int_0^x\int_0^y\left(\int_0^s\int_0^t\left(\int_0^\sigma\int_0^\tau h(s,t,\sigma,\tau,m,n)g(u(m,n))dndm\right)d\tau d\sigma\right)dtds, \tag{2.11}$$

for $(x,y)\in\Delta$, *then for* $0\le x\le x_2$, $0\le y\le y_2$; x, $x_2\in I$, y, $y_2\in J$,

$$u(x,y)\le G^{-1}\left[G(c)+\int_0^x\int_0^y Q(s,t)dtds\right], \tag{2.12}$$

where $Q(x,y)$ *is given by* (2.10), G, G^{-1} *are as in Theorem* 1, *part* (a$_2$) *and* $x_2\in I$, $y_2\in J$ *are chosen so that*

$$G(c)+\int_0^x\int_0^y Q(s,t)dtds\in\mathrm{Dom}(G^{-1}),$$

for all $(x,y)\in\Delta$ *such that* $0\le x\le x_2$, $0\le y\le y_2$.

Theorem 3. *Let* $u(x,y)\in C(\Delta,R_+)$ *and* $c\ge 0$ *be a constant.*

(c$_1$) *Let* $k(x,y,s,t)$, $e(x,y,s,t)\in C(D,R_+)$, $h(x,y,s,t,\sigma,\tau)\in C(E,R_+)$ *be nondecreasing in* $(x,y)\in\Delta$ *for fixed* $(s,t)\in\Delta$, $(s,t,\sigma,\tau)\in\Delta^2$ *and suppose that*

$$\begin{aligned}u(x,y)\le c&+\int_0^x\int_0^y k(x,y,s,t)u(s,t)dtds\\&+\int_0^x\int_0^y\left(\int_0^s\int_0^t h(x,y,s,t,\sigma,\tau)u(\sigma,\tau)d\tau d\sigma\right)dtds\\&+\int_0^\alpha\int_0^\beta e(x,y,s,t)u(s,t)dtds,\end{aligned}\tag{2.13}$$

for $(x,y)\in\Delta$. *If*

$$\begin{aligned}p(x,y)=&\int_0^\alpha\int_0^\beta e(x,y,s,t)\\&\times\exp\left(\int_0^s\int_0^t\left[k(s,t,m,n)+\int_0^m\int_0^n h(s,t,m,n,\sigma,\tau)d\tau d\sigma\right]dndm\right)dtds<1,\end{aligned}\tag{2.14}$$

for $(x,y)\in\Delta$, *then*

$$u(x,y)\le\frac{c}{1-p(x,y)}\exp\left(\int_0^x\int_0^y\left[k(x,y,m,n)+\int_0^m\int_0^n h(x,y,m,n,\sigma,\tau)d\tau d\sigma\right]dndm\right), \tag{2.15}$$

for $(x,y)\in\Delta$.

(c$_2$) *Let* $a(x,y)$, $b(x,y)\in C(\Delta,R_+)$, $k(x,y,s,t)\in C(D,R_+)$, $h(x,y,s,t,\sigma,\tau)\in C(E,R_+)$ *and suppose that*

$$\begin{aligned}u(x,y)\le c&+\int_0^x\int_0^y a(s,t)u(s,t)dtds+\int_0^x\int_0^y\left(\int_0^s\int_0^t k(s,t,\sigma,\tau)u(\sigma,\tau)d\tau d\sigma\right)dtds\\&+\int_0^x\int_0^y\left(\int_0^s\int_0^t\left(\int_0^\sigma\int_0^\tau h(s,t,\sigma,\tau,m,n)u(m,n)dndm\right)d\tau d\sigma\right)dtds\\&+\int_0^\alpha\int_0^\beta b(s,t)u(s,t)dtds,\end{aligned}\tag{2.16}$$

for $(x, y) \in \Delta$. *If*

$$q = \int_0^\alpha \int_0^\beta b(s,t) \exp\left(\int_0^s \int_0^t Q(\xi,\eta) d\eta d\xi\right) dtds < 1, \tag{2.17}$$

where $Q(x, y)$ *is given by* (2.10), *then*

$$u(x,y) \leq \frac{c}{1-q} \exp\left(\int_0^x \int_0^y Q(s,t) dtds\right), \tag{2.18}$$

for $(x, y) \in \Delta$.

3. Proofs of Theorems 1-3

(a_1) We frist assume that $c > 0$ and define a function $z(x, y)$ by the right hand side of (2.1). Then $z(x, y) > 0$, $u(x, y) \leq z(x, y)$, $z(0, y) = z(x, 0) = c$ and $z(x, y)$ is nondecreasing in both the variables $(x, y) \in \Delta$. It is easy to observe that (see [7, p.328])

$$z_{xy}(x,y) \leq [A(x,y) + B(x,y)] z(x,y), \tag{3.1}$$

where $A(x, y)$, $B(x, y)$ are given by (2.3), (2.4). Now by following the proof of Theorem 4.2.1 given in [7], from (3.1) we get

$$z(x,y) \leq c \exp\left(\int_0^x \int_0^y [A(m,n) + B(m,n)] dndm\right), \tag{3.2}$$

for $(x, y) \in \Delta$. Using (3.2) in $u(x, y) \leq z(x, y)$, we get the required inequality in (2.2).

If $c \geq 0$ we carry out the above procedure with $c + \epsilon$ instead of c, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (2.2).

(a_2) We note that since $g'(u) \geq 0$ on R_+, the function $g(u)$ is monotone increasing on $(0, \infty)$. Assume that $c > 0$ and define a function $z(x, y)$ by the right hand side of (2.5). Then $z(x, y) > 0$, $u(x, y) \leq z(x, y)$, $z(0, y) = z(x, 0) = c$ and $z(x, y)$ is nondecreasing in both the variables $(x, y) \in \Delta$. It is easy to see that

$$z_{xy}(x,y) \leq [A(x,y) + B(x,y)] g(z(x,y)), \tag{3.3}$$

where $A(x, y)$, $B(x, y)$ are given by (2.3), (2.4). The remaining proof can be completed by following the proof of Theorem 5.2.1 given in [7]. The proof of the case when $c \geq 0$ follows as mentioned in the proof of (a_1).

(b_1) Let $c > 0$ and define a function $z(x, y)$ by the right hand side of (2.8). Then $z(x, y) > 0$, $u(x, y) \leq z(x, y)$, $z(0, y) = z(x, 0) = c$ and $z(x, y)$ is nondecreasing in both the variables $(x, y) \in \Delta$ and

$$\begin{aligned} z_{xy}(x,y) &= a(x,y)u(x,y) + \int_0^x \int_0^y k(x,y,\sigma,\tau) u(\sigma,\tau) d\tau d\sigma \\ &\quad + \int_0^x \int_0^y \left(\int_0^\sigma \int_0^\tau h(x,y,\sigma,\tau,m,n) u(m,n) dndm\right) d\tau d\sigma \\ &\leq Q(x,y) z(x,y) \end{aligned} \tag{3.4}$$

where $Q(x, y)$ is given by (2.10). Following the proof of Theorem 4.2.1 given in [7], from (3.4) we get

$$z(x, y) \leq c \exp \left(\int_0^x \int_0^y Q(s, t) dt ds \right). \tag{3.5}$$

Using (3.5) in $u(x, y) \leq z(x, y)$ we get the desired inequality in (2.9). The case when $c \geq 0$ follows as noted in the proof of (a_1).

(b_2) The proof can be completed by following the proof of (b_1) and closely looking at the proof of Theorem 5.2.1 given in [7]. Here we leave the details to the reader.

(c_1) Frist assume that $c > 0$ and fix any arbitrary $(X, Y) \in \Delta$. Then for $0 \leq x \leq X$, $0 \leq y \leq Y$ we have

$$\begin{aligned} u(x, y) \leq c &+ \int_0^x \int_0^y k(X, Y, s, t) u(s, t) dt ds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t h(X, Y, s, t, \sigma, \tau) u(\sigma, \tau) d\tau d\sigma \right) dt ds \\ &+ \int_0^\alpha \int_0^\beta e(X, Y, s, t) u(s, t) dt ds. \end{aligned} \tag{3.6}$$

Let

$$d = c + \int_0^\alpha \int_0^\beta e(X, Y, s, t) u(s, t) dt ds, \tag{3.7}$$

then (3.6) can be restated as

$$\begin{aligned} u(x, y) \leq d &+ \int_0^x \int_0^y k(X, Y, s, t) u(s, t) dt ds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t h(X, Y, s, t, \sigma, \tau) u(\sigma, \tau) d\tau d\sigma \right) dt ds, \end{aligned} \tag{3.8}$$

for $0 \leq x \leq X$, $0 \leq y \leq Y$. Define a function $z(x, y)$ by the right hand side of (3.8). Then $z(x, y) > 0$, $u(x, y) \leq z(x, y)$, $z(0, y) = z(x, 0) = d$, $z(x, y)$ is nondecreasing in both the variables x, y lying in $0 \leq x \leq X$, $0 \leq y \leq Y$ and

$$\begin{aligned} z_{xy}(x, y) &= k(X, Y, x, y) u(x, y) + \int_0^x \int_0^y h(X, Y, x, y, \sigma, \tau) u(\sigma, \tau) d\tau d\sigma \\ &\leq \left[k(X, Y, x, y) + \int_0^x \int_0^y h(X, Y, x, y, \sigma, \tau) d\tau d\sigma \right] z(x, y). \end{aligned} \tag{3.9}$$

Now by following the proof of Theorem 4.2.1 given in [7], from (3.9) we get

$$z(x, y) \leq d \exp \left(\int_0^x \int_0^y \left[k(X, Y, m, n) + \int_0^m \int_0^n h(X, Y, m, n, \sigma, \tau) d\tau d\sigma \right] dn dm \right). \tag{3.10}$$

for $0 \le x \le X$, $0 \le y \le Y$. Since $(X, Y) \in \Delta$ is arbitrary, from (3.10), (3.7) with (X, Y) replaced by (x, y) and $u(x, y) \le z(x, y)$ we have

$$u(x,y) \le d \exp\left(\int_0^x \int_0^y \left[k(x,y,m,n) + \int_0^m \int_0^n h(x,y,m,n,\sigma,\tau) d\tau d\sigma\right] dndm\right), \tag{3.11}$$

for $(x, y) \in \Delta$, where

$$d = c + \int_0^\alpha \int_0^\beta e(x,y,s,t)u(s,t)dtds, \tag{3.12}$$

for $(x, y) \in \Delta$. Using (3.11) in the integrand on the right hand side of (3.12) and in view of (2.14) we have

$$d \le \frac{c}{1 - p(x,y)}. \tag{3.13}$$

Using (3.13) in (3.11), we get the required inequality in (2.15). The proof of the case when $c \ge 0$ follows as noted in the proof of (a_1).

(c_2) Let $c > 0$ and denote

$$d' = c + \int_0^\alpha \int_0^\beta b(s,t)u(s,t)dtds. \tag{3.14}$$

Then (2.16) can be restated as

$$\begin{aligned} u(x,y) \le d' &+ \int_0^x \int_0^y a(s,t)u(s,t)dtds + \int_0^x \int_0^y \left(\int_0^s \int_0^t k(s,t,\sigma,\tau)u(\sigma,\tau)d\tau d\sigma\right) dtds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t \left(\int_0^\sigma \int_0^\tau h(s,t,\sigma,\tau,m,n)u(m,n)dndm\right) d\tau d\sigma\right) dtds. \end{aligned} \tag{3.15}$$

Define a function $z(x, y)$ by the right hand side of (3.15). Then $z(x, y) > 0$, $u(x, y) \le z(x, y)$, $z(0, y) = z(x, 0) = d'$, $z(x, y)$ is nondecreasing in both the variables $(x, y) \in \Delta$ and

$$\begin{aligned} z_{xy}(x,y) &= a(x,y)u(x,y) + \int_0^x \int_0^y k(x,y,\sigma,\tau)u(\sigma,\tau)d\tau d\sigma \\ &\quad + \int_0^x \int_0^y \left(\int_0^\sigma \int_0^\tau h(x,y,\sigma,\tau,m,n)u(m,n)dndm\right) d\tau d\sigma \\ &\le Q(x,y)z(x,y). \end{aligned} \tag{3.16}$$

The rest of the proof can be completed by following the proof of Theorem 4.2.1 given in [7] and closely looking at the proof of (c_1) given above.

4. Applications

In this section, we present some applications of the inequality given in Theorem 3, part (c_1) to study certain properties of solutions of the Volterra-Fredholm integral equation of the form

$$\begin{aligned} z(x,y) = f(x,y) &+ \int_0^x \int_0^y F(x,y,s,t,z(s,t))dtds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t H(x,y,s,t,\sigma,\tau,z(\sigma,\tau))d\tau d\sigma \right) dtds \\ &+ \int_0^\alpha \int_0^\beta L(x,y,s,t,z(s,t))dtds, \end{aligned} \tag{4.1}$$

where $z,\ f \in C(\Delta, R)$, $F, L \in C(D \times R, R)$, $H \in C(E \times R, R)$. Here we note that the existence proofs for the solutions of equation (4.1) show either that the operator T defined by the right hand side of (4.1) is a contraction (in which case one also has uniqueness) or T is compact and continuous on a suitable subspace of the space of continuous functions (see also [5, 6]).

The following theorem gives the explicit bound on the solution of equation (4.1).

Theorem 4. *Suppose that the functions f, F, H, L satisfy the conditions*

$$|f(x,y)| \le c, \tag{4.2}$$

$$|F(x,y,s,t,z)| \le k(x,y,s,t)|z|, \tag{4.3}$$

$$|H(x,y,s,t,\sigma,\tau,z)| \le h(x,y,s,t,\sigma,\tau)|z|, \tag{4.4}$$

$$|L(x,y,s,t,z)| \le e(x,y,s,t)|z|, \tag{4.5}$$

where c, k, h, e are as in Theorem 3, part (c_1). Let $p(x,y)$ be as in (2.14). If $z(x,y)$ is a solution of equation (4.1) on Δ, then

$$\begin{aligned} |z(x,y)| \le & \frac{c}{1-p(x,y)} \\ & \times \exp\left(\int_0^x \int_0^y \left[k(x,y,m,n) + \int_0^m \int_0^n h(x,y,m,n,\sigma,\tau)d\tau d\sigma \right] dndm \right), \end{aligned} \tag{4.6}$$

for $(x,y) \in \Delta$.

Proof. The solution $z(x,y)$ satisfies the equation (4.1). Using (4.2)−(4.5) in (4.1) we have

$$\begin{aligned} |z(x,y)| \le c &+ \int_0^x \int_0^y k(x,y,s,t)|z(s,t)|dtds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t h(x,y,s,t,\sigma,\tau)|z(\sigma,\tau)|d\tau d\sigma \right) dtds \\ &+ \int_0^\alpha \int_0^\beta e(x,y,s,t)|z(s,t)|dtds. \end{aligned} \tag{4.7}$$

Now an application of the inquality given in Theroem 3, part (c_1) to (4.7) yields the bound in (4.6).

The next result deals with the uniqueness of solutions of equation (4.1).

Theorem 5. *Suppose that the functions F, H, L in* (4.1) *satisfy the conditions*

$$|F(x,y,s,t,z) - F(x,y,s,t,\bar{z})| \leq k(x,y,s,t)|z-\bar{z}|, \tag{4.8}$$

$$|H(x,y,s,t,\sigma,\tau,z) - H(x,y,s,t,\sigma,\tau,\bar{z})| \leq h(x,y,s,t,\sigma,\tau)|z-\bar{z}|, \tag{4.9}$$

$$|L(x,y,s,t,z) - L(x,y,s,t,\bar{z})| \leq e(x,y,s,t)|z-\bar{z}|, \tag{4.10}$$

where k, h, e are as in Theorem 3, *part* (c_1). *Let $p(x,y)$ be as in* (2.14). *Then the equation* (4.1) *has at most one solution on* Δ.

Proof. Let $z(x,y)$ and $\bar{z}(x,y)$ be two solutions of equation (4.1) on Δ. Using the facts that $z(x,y)$ and $\bar{z}(x,y)$ satisfy the equation (4.1) and the conditions (4.8)–(4.10) we have

$$\begin{aligned}|z(x,y) - \bar{z}(x,y)| \leq & \int_0^x \int_0^y k(x,y,s,t)|z(s,t) - \bar{z}(s,t)|dtds \\ & + \int_0^x \int_0^y \left(\int_0^s \int_0^t h(x,y,s,t,\sigma,\tau)|z(\sigma,\tau) - \bar{z}(\sigma,\tau)|d\tau d\sigma \right) dtds \\ & + \int_0^\alpha \int_0^\beta e(x,y,s,t)|z(s,t) - \bar{z}(s,t)|dtds. \end{aligned} \tag{4.11}$$

Now a suitable application of Theorem 3, part (c_1) (when $c=0$) to (4.11) yields

$$|z(x,y) - \bar{z}(x,y)| \leq 0.$$

Therefore $z(x,y) = \bar{z}(x,y)$, i.e. there is at most one solution to the equation (4.1) on Δ.

The next theorem deals with the continuous dependence of solutions of equation (4.1) on the fucntions involved on the right hand side of equation (4.1).

Consider the equation (4.1) and the following equation

$$\begin{aligned} w(x,y) = & \bar{f}(x,y) + \int_0^x \int_0^y \bar{F}(x,y,s,t,w(s,t))dtds \\ & + \int_0^x \int_0^y \left(\int_0^s \int_0^t \bar{H}(x,y,s,t,\sigma,\tau,w(\sigma,\tau))d\tau d\sigma \right) dtds \\ & + \int_0^\alpha \int_0^\beta \bar{L}(x,y,s,t,w(s,t))dtds \end{aligned} \tag{4.12}$$

where w, $\bar{f} \in C(\Delta,R)$, $\bar{F}$, $\bar{L} \in C(D \times R, R)$, $\bar{H} \in C(E \times R, R)$.

Theorem 6. *Suppose that the functions F, H, L in equation* (4.1) *satisfy the conditions* (4.8), (4.9), (4.10) *in Theorem* 5 *and further assume that*

$$|f(x,y) - \bar{f}(x,y)| \leq \epsilon, \tag{4.13}$$

$$\int_0^x \int_0^y |F(x,y,s,t,w(s,t)) - \bar{F}(x,y,s,t,w(s,t))| dtds \le \epsilon, \tag{4.14}$$

$$\int_0^x \int_0^y \left(\int_0^s \int_0^t |H(x,y,s,t,\sigma,\tau,w(s,t)) - \bar{H}(x,y,s,t,\sigma,\tau,w(s,t))| d\tau d\sigma \right) dtds \le \epsilon, \tag{4.15}$$

$$\int_0^\alpha \int_0^\beta |L(x,y,s,t,w(s,t)) - \bar{L}(x,y,s,t,w(s,t))| dtds \le \epsilon, \tag{4.16}$$

where $\epsilon > 0$ is an arbitrary small constant. Let $p(x,y)$ be as in (2.14)*. Then the solution of equation* (4.1) *depends continuously on the functions involved on the right hand side of equation* (4.1)*.*

Proof. Let $z(x,y)$ and $w(x,y)$ be the solutions of equations (4.1) and (4.12) respectively. Then we have

$$\begin{aligned} & z(x,y) - w(x,y) \\ &= f(x,y) - \bar{f}(x,y) + \int_0^x \int_0^y \{F(x,y,s,t,z(s,t)) - F(x,y,s,t,w(s,t))\} dtds \\ &\quad + \int_0^x \int_0^y \{F(x,y,s,t,w(s,t)) - \bar{F}(x,y,s,t,w(s,t))\} dtds \\ &\quad + \int_0^x \int_0^y \left(\int_0^s \int_0^t \{H(x,y,s,t,\sigma,\tau,z(\sigma,\tau)) - H(x,y,s,t,\sigma,\tau,w(\sigma,\tau))\} d\sigma d\tau \right) dtds \\ &\quad + \int_0^x \int_0^y \left(\int_0^s \int_0^t \{H(x,y,s,t,\sigma,\tau,w(\sigma,\tau)) - \bar{H}(x,y,s,t,\sigma,\tau,w(\sigma,\tau))\} d\sigma d\tau \right) dtds \\ &\quad + \int_0^\alpha \int_0^\beta \{L(x,y,s,t,z(s,t)) - L(x,y,s,t,w(s,t))\} dtds \\ &\quad + \int_0^\alpha \int_0^\beta \{L(x,y,s,t,w(s,t)) - \bar{L}(x,y,s,t,w(s,t))\} dtds. \end{aligned} \tag{4.17}$$

Using (4.8)–(4.10) and (4.13)–(4.16) in (4.17) we get

$$\begin{aligned} |z(x,y) - w(x,y)| \le 4\epsilon &+ \int_0^x \int_0^y k(x,y,s,t) |z(s,t) - w(s,t)| dtds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t h(x,y,s,t,\sigma,\tau) |z(\sigma,\tau) - w(\sigma,\tau)| d\tau d\sigma \right) dtds \\ &+ \int_0^\alpha \int_0^\beta e(x,y,s,t) |z(s,t) - w(s,t)| dtds. \end{aligned} \tag{4.18}$$

Now an application of Theorem 3, part (c_1) to (4.18) yields

$$\begin{aligned} & |z(x,y) - w(x,y)| \\ &\le 4\epsilon \left[\frac{1}{1 - p(x,y)} \exp\left(\int_0^x \int_0^y \left[k(x,y,m,n) \int_0^m \int_0^n h(x,y,m,n,\sigma,\tau) d\tau d\sigma \right] dndm \right) \right], \end{aligned} \tag{4.19}$$

for $(x,y) \in \Delta$. On the compact set, the quantity in first square bracket in (4.19) is bounded by some positive constant, say M. Therefore $|z(x,y)-w(x,y)| \leq 4 \in M$ on the set, so the solution of equation (4.1) depends continuously on the functions involved on the right hand side of equation (4.1). If $\epsilon \to 0$, then $|z(x,y)-w(x,y)| \to 0$ on the set.

We next consider the following Volterra-Fredholm integral equations

$$\begin{aligned} z(x,y) = f(x,y) &+ \int_0^x \int_0^y F(x,y,s,t,z(s,t),\mu)dtds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t H(x,y,s,t,\sigma,\tau,z(\sigma,\tau),\mu)d\tau d\sigma \right) dtds \\ &+ \int_0^\alpha \int_0^\beta L(x,y,s,t,z(s,t),\mu)dtds, \end{aligned} \tag{4.20}$$

$$\begin{aligned} z(x,y) = f(x,y) &+ \int_0^x \int_0^y F(x,y,s,t,z(s,t),\mu_0)dtds \\ &+ \int_0^x \int_0^y \left(\int_0^s \int_0^t H(x,y,s,t,\sigma,\tau,z(\sigma,\tau),\mu_0)d\tau d\sigma \right) dtds \\ &+ \int_0^\alpha \int_0^\beta L(x,y,s,t,z(s,t),\mu_0)dtds, \end{aligned} \tag{4.21}$$

where $z,\ f \in C(\Delta, R),\ F,\ L \in C(D \times R^2, R),\ H \in C(E \times R^2, R)$ and $\mu,\ \mu_0$ are real parameters.

The following theorem shows the dependency of solutions of equations (4.20) and (4.21) on parameters.

Theorem 7. *Suppose that*

$$|F(x,y,s,t,z,\mu) - F(x,y,s,t,\bar{z},\mu)| \leq k(x,y,s,t)|z-\bar{z}|, \tag{4.22}$$

$$|F(x,y,s,t,z,\mu) - F(x,y,s,t,z,\mu_0)| \leq r_1(x,y,s,t)|\mu-\mu_0|, \tag{4.23}$$

$$|H(x,y,s,t,\sigma,\tau,z,\mu) - H(x,y,s,t,\sigma,\tau,\bar{z},\mu)| \leq h(x,y,s,t,\sigma,\tau)|z-\bar{z}|, \tag{4.24}$$

$$|H(x,y,s,t,\sigma,\tau,z,\mu) - H(x,y,s,t,\sigma,\tau,z,\mu_0)| \leq r_2(x,y,s,t,\sigma,\tau)|\mu-\mu_0|, \tag{4.25}$$

$$|L(x,y,s,t,z,\mu) - L(x,y,s,t,\bar{z},\mu)| \leq e(x,y,s,t)|z-\bar{z}|, \tag{4.26}$$

$$|L(x,y,s,t,z,\mu) - L(x,y,s,t,z,\mu_0)| \leq r_3(x,y,s,t)|\mu-\mu_0|, \tag{4.27}$$

where k, h, e are as in Theorem 3*, part* (c_1) *and $r_1,\ r_3 \in C(D, R_+)$, $r_2 \in C(E, R_+)$ are such that*

$$\int_0^x \int_0^y r_1(x,y,s,t)dtds \leq k_1, \tag{4.28}$$

$$\int_0^x \int_0^y \left(\int_0^s \int_0^t r_2(x,y,s,t,\sigma,\tau)d\tau d\sigma \right) dtds \leq k_2, \tag{4.29}$$

$$\int_0^\alpha \int_0^\beta r_3(x,y,s,t)dtds \leq k_3, \tag{4.30}$$

where k_1, k_2, k_3 are positive constants. Let $p(x,y)$ be given by (2.14)*. Let $z_1(x,y)$ and $z_2(x,y)$ be the solutions of equations* (4.20) *and* (4.21) *respectively. Then*

$$\begin{aligned}&|z_1(x,y)-z_2(x,y)|\\&\le \frac{(k_1+k_2+k_3)|\mu-\mu_0|}{1-p(x,y)}\\&\times\exp\left(\int_0^x\int_0^y\left[k(x,y,m,n)+\int_0^m\int_0^n h(x,y,m,n,\sigma,\tau)d\tau d\sigma\right]dndm\right),\end{aligned} \tag{4.31}$$

for $(x,y)\in\Delta$.

Proof. Let $z(x,y)=z_1(x,y)-z_2(x,y)$, $(x,y)\in\Delta$. Then

$$\begin{aligned}z(x,y)=&\int_0^x\int_0^y\{F(x,y,s,t,z_1(s,t),\mu)-F(x,y,s,t,z_2(s,t),\mu)\}dtds\\&+\int_0^x\int_0^y\{F(x,y,s,t,z_2(s,t),\mu)-F(x,y,s,t,z_2(s,t),\mu_0)\}dtds\\&+\int_0^x\int_0^y\left(\int_0^s\int_0^t\{H(x,y,s,t,\sigma,\tau,z_1(\sigma,\tau),\mu)\right.\\&\left.-H(x,y,s,t,\sigma,\tau,z_2(\sigma,\tau),\mu)\}d\tau d\sigma\right)dtds\\&+\int_0^x\int_0^y\left(\int_0^s\int_0^t\{H(x,y,s,t,\sigma,\tau,z_2(\sigma,\tau),\mu)\right.\\&\left.-H(x,y,s,t,\sigma,\tau,z_2(\sigma,\tau),\mu_0)\}d\tau d\sigma\right)dtds\\&+\int_0^\alpha\int_0^\beta\{L(x,y,s,t,z_1(s,t),\mu)-L(x,y,s,t,z_2(s,t),\mu)\}dtds\\&+\int_0^\alpha\int_0^\beta\{L(x,y,s,t,z_2(s,t),\mu)-L(x,y,s,t,z_2(s,t),\mu_0)\}dtds.\end{aligned} \tag{4.32}$$

Using (4.22)−(4.30) in (4.32) we get

$$\begin{aligned}|z(x,y)|\le(k_1+k_2+k_3)|\mu-\mu_0|&+\int_0^x\int_0^y k(x,y,s,t)|z(s,t)|dtds\\&+\int_0^x\int_0^y\left(\int_0^s\int_0^t h(x,y,s,t,\sigma,\tau)|z(\sigma,\tau)|d\tau d\sigma\right)dtds\\&+\int_0^\alpha\int_0^\beta e(x,y,s,t)|z(s,t)|dtds.\end{aligned} \tag{4.33}$$

Now a suitable application of Theorem 3, part (c_1) to (4.33) yields (4.31), which shows the dependency of solutions of equations (4.20) and (4.21) on parameters.

In conclusion, we note that the results given in Theorems 1−3 can be generalized to obtain explicit bounds on integral inequalities involving many iterated double integrals and also to the functions with more than two independent variables. We leave the formulation of such results to the reader to fill in where needed. Various other applications of the inequalities given here is left to another work.

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