AN INVERSE SPECTRAL PROBLEM FOR NON-SELFADJOINT STURM-LIOUVILLE OPERATORS WITH NONSEPARATED BOUNDARY CONDITIONS

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Abstract. Non-selfadjoint Sturm-Liouville operators on a finite interval with nonseparated boundary conditions are studied. We establish properties of the spectral characteristics and investigate an inverse problem of recovering the operators from their spectral data. For this inverse problem we prove a uniqueness theorem and provide a procedure for constructing the solution.

1. Introduction

Consider the boundary value problem $L = L(q, h, \alpha, \beta, T)$ of the form

$$- y'' + q(x)y = \lambda y, \quad x \in [0, T], \tag{1}$$

$$y(0) = \alpha y(T), \quad y'(0) - h y(0) = \beta y'(T), \tag{2}$$

where $q(x) \in L(0, T)$ is a complex-valued function, α, β, h are complex numbers, and $\alpha + \beta \neq 0$, $\alpha\beta \neq 0$. We study a nonlinear inverse problem of recovering L from its spectral data. For this inverse problem we prove a uniqueness theorem and provide a procedure for constructing the solution. Similar results are valid for another type of nonseparated boundary conditions: $y'(0) - h y(0) + \alpha y(T) = y'(T) + H y(T) - \beta y(0) = 0$.

Inverse spectral problems often appear in mathematics as well as in applications [1, 2, 3, 4]. For *selfadjoint* Sturm-Liouville operators with *separated* boundary conditions, inverse spectral problems have been studied fairly completely (see the monographs [1, 2, 3, 4] and the references therein). Inverse problems for *nonseparated* boundary conditions in the *selfadjoint* case were investigated in [5]-[13] and other works. Non-selfadjoint case is more difficult for studying. Inverse problems for non-selfadjoint Sturm-Liouville operators with *separated* boundary conditions were investigated in [4, 14, 15]. In [16] the structure of the spectrum for the non-selfadjoint periodic case is investigated. Inverse problems for non-selfadjoint

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boundary value problem L have not been studied yet. We note that the inverse problem, considered here, appears in the inverse problem theory for differential operators on spatial networks with cycles (see [17, 18, 19, 20, 21]) which have many applications in natural sciences and engineering.

Some words about the structure of the paper. In section 2 properties of the spectrum are established. The Weyl-type function and the corresponding Weyl sequence are introduced and investigated in section 3. Section 4 is devoted to the solution of the inverse spectral problem for L. The main result is formulated in theorem 3. In section 5 we study the periodic boundary value problem which is a particular case of L. For convenience of readers, we formulate there the results from [8] (published in 1981 only in Russian) where necessary and sufficient conditions for the solvability of the inverse problem were obtained for the selfadjoint periodic case.

2. Properties of the spetrum

Let $C(x,\lambda)$, $S(x,\lambda)$ and $\varphi(x,\lambda)$ be solutions of equation (1) under the initial conditions

$$C(0,\lambda) = S'(0,\lambda) = \varphi(0,\lambda) = 1,\ C'(0,\lambda) = S(0,\lambda) = 0,\ \varphi'(0,\lambda) = h.$$

Clearly, $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$. Let $\lambda = \rho^2$, $\tau = \operatorname{Im}\rho$, $q_1(x) = \frac{1}{2}\int_0^x q(t)\,dt$, $q_1 = q_1(T)$. It is known (see, for example [3]) that for $|\rho| \to \infty$,

$$C(x,\lambda) = \cos\rho x + q_1(x)\frac{\sin\rho x}{\rho} + o\left(\frac{e^{|\tau|x}}{\rho}\right),\quad C'(x,\lambda) = -\rho\sin\rho x + q_1(x)\cos\rho x + o(e^{|\tau|x}),$$

$$S(x,\lambda) = \frac{\sin\rho x}{\rho} - q_1(x)\frac{\cos\rho x}{\rho^2} + o\left(\frac{e^{|\tau|x}}{\rho^2}\right),\quad S'(x,\lambda) = \cos\rho x + q_1(x)\frac{\sin\rho x}{\rho} + o\left(\frac{e^{|\tau|x}}{\rho}\right).$$

The function $\Delta(\lambda) = \alpha\varphi(T,\lambda) + \beta S'(T,\lambda) - (1 + \alpha\beta)$ is entire in λ of order $1/2$, and the set of its zeros $\Lambda = \{\lambda_n\}_{n\geq 0}$ (counting with multiplicities) coincides with the eigenvalues of L. The function $\Delta(\lambda)$ is called the characteristic function for L. For $|\rho| \to \infty$,

$$\Delta(\lambda) = (\alpha + \beta)\left(\cos\rho T - \gamma + \gamma_1\frac{\sin\rho T}{\rho} + o\left(\frac{e^{|\tau|T}}{\rho}\right)\right), \tag{3}$$

where $\gamma = \frac{\alpha\beta+1}{\alpha+\beta}$ and $\gamma_1 = q_1 + \frac{h\alpha}{\alpha+\beta}$. Let $\{\lambda_n^0\}_{n\geq 0}$ be the eigenvalues of $L^0 = L(0,0,\alpha,\beta,T)$ with $q(x) = 0$, $h = 0$. Then the characteristic function of L^0 has the form

$$\Delta^0(\lambda) = (\alpha + \beta)(\cos\rho T - \gamma) = \frac{\alpha + \beta}{2}\left(e^{i\rho T} + e^{-i\rho T} - 2\gamma\right). \tag{4}$$

Let $z = e^{i\rho T}$, and let $z_{1,2} = \gamma \pm \sqrt{\gamma^2 - 1}$ be the roots of the equation $z^2 - 2\gamma z + 1 = 0$. Then $\lambda_n^0 = (\rho_n^0)^2$, $\{\rho_n^0\} = \{\rho_{n,1}^0\} \cup \{\rho_{n,2}^0\}$,

$$\rho_{n,j}^0 = \frac{2n\pi}{T} - \frac{i}{T}\ln z_j,\quad j = 1,2.$$

We agree that if $z = |z|e^{i\xi}$, $\xi \in [0, 2\pi)$ then $\ln z = \ln|z| + i\xi$, $\sqrt{z} = |z|^{1/2}e^{i\xi/2}$.

Examples. (1) Let $\gamma = 1$. Then $z_1 = z_2 = 1$, $\ln z_j = 0$,

$$\Delta^0(\lambda) = (\alpha + \beta)(\cos\rho T - 1), \quad \rho^0_{n,1} = \frac{2n\pi}{T}, \ n \geq 0, \quad \rho^0_{n,2} = \frac{2n\pi}{T}, \ n \geq 1.$$

(2) Let $\gamma = -1$. Then $z_1 = z_2 = -1$, $\ln z_j = \pi i$, $\rho^0_{n,1} = \rho^0_{n,2} = \dfrac{(2n+1)\pi}{T}$, $n \geq 0$.

(3) If $\gamma \neq \pm 1$, then all eigenvalues $\{\lambda^0_n\}_{n\geq 0}$ are simple. Let, for example, $\gamma = 1/2$. Then

$$\rho^0_{n,1} = \frac{1}{T}\left(\frac{\pi}{3} + 2n\pi\right), \ n \geq 0, \quad \rho^0_{n,2} = \frac{1}{T}\left(-\frac{\pi}{3} + 2n\pi\right), \ n \geq 1.$$

If $\gamma = 0$, then $z_{1,2} = \pm i$, $\ln z_1 = \pi i/2$, $\ln z_2 = 3\pi i/2$, $\rho^0_n = (n + 1/2)\frac{\pi}{T}$, $n \geq 0$.

Using (3), by the known method (see [22]) one gets $\lambda_n = \rho^2_n$, $\{\rho_n\} = \{\rho_{n,1}\} \cup \{\rho_{n,2}\}$,

$$\rho_{n,j} = \rho^0_{n,j} + \varepsilon_{n,j}, \quad \varepsilon_{n,j} = o(1), \quad n \to \infty. \tag{5}$$

More precisely (see [23]),

$$\varepsilon_{n,j} = \frac{\gamma_1}{2\pi n} + o\left(\frac{1}{n}\right), \quad \text{if} \quad \gamma \neq \pm 1,$$

$$\varepsilon_{n,j} = \frac{1}{2\pi n}\left(\gamma_1 + (-1)^j h/2\right) + o\left(\frac{1}{n}\right), \quad \text{if} \quad \gamma = \pm 1, \ \alpha = \beta,$$

$$\varepsilon_{n,j} = o\left(\frac{1}{\sqrt{n}}\right), \quad \text{if} \quad \gamma = \pm 1, \ \alpha \neq \beta.$$

Theorem 1. (i) *Let $\gamma \neq 1$ (i.e. $\Delta^0(0) \neq 0$). Then*

$$\Delta(\lambda) = (\alpha + \beta - 1 - \alpha\beta)\prod_{n=0}^{\infty} \frac{\lambda_n - \lambda}{\lambda^0_n}, \quad \lambda^0_n \neq 0. \tag{6}$$

(ii) *Let $\gamma = 1$ (i.e. $\Delta^0(0) = 0$). Then*

$$\Delta(\lambda) = -\frac{(\alpha + \beta)T^2}{2}(\lambda - \lambda_0)\prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{\lambda^0_n}, \quad \lambda^0_{2n} = \lambda^0_{2n-1} = \left(\frac{2\pi n}{T}\right)^2. \tag{7}$$

Proof. It follows from (4) that

$$\dot{\Delta}^0(\lambda) = -(\alpha + \beta)T\frac{\sin\rho T}{2\rho}, \quad \dot{\Delta}^0(\lambda) := \frac{d}{d\lambda}\Delta^0(\lambda) = \frac{1}{2\rho}\frac{d}{d\rho}\Delta^0(\lambda),$$

and consequently, $\Delta^0(0) = \alpha + \beta - 1 - \alpha\beta$, $\dot{\Delta}^0(0) = -(\alpha + \beta)T^2/2 \neq 0$. Using Hadamard's factorization theorem we get

$$\Delta^0(\lambda) = A^0\prod_{n=0}^{\infty} \frac{\lambda^0_n - \lambda}{\lambda^0_n}, \ A^0 = \alpha + \beta - 1 - \alpha\beta, \ \lambda^0_n \neq 0, \qquad\qquad \text{if} \quad \gamma \neq 1,$$

$$\Delta^0(\lambda) = B^0 \lambda \prod_{n=1}^{\infty} \frac{\lambda_n^0 - \lambda}{\lambda_n^0}, \ B^0 = -\frac{(\alpha + \beta)T^2}{2}, \ \lambda_{2n}^0 = \lambda_{2n-1}^0 = \left(\frac{2\pi n}{T}\right)^2, \quad \text{if} \quad \gamma = 1.$$

Let for definiteness, $\gamma \neq 1$, $\Delta(0) \neq 0$ (other cases are treated similarly). Then, by Hadamard's factorization theorem,

$$\Delta(\lambda) = A \prod_{n=0}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n},$$

where $A \neq 0$ is a constant. Therefore,

$$\frac{\Delta(\lambda)}{\Delta^0(\lambda)} = \frac{A}{A^0} \prod_{n=0}^{\infty} \frac{\lambda_n^0}{\lambda_n} \prod_{n=0}^{\infty} \frac{\lambda - \lambda_n}{\lambda - \lambda_n^0} = \frac{A}{A^0} \prod_{n=0}^{\infty} \frac{\lambda_n^0}{\lambda_n} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda_n - \lambda_n^0}{\lambda - \lambda_n^0}\right).$$

By virtue of (3)−(5),

$$\lim_{\lambda \to -\infty} \frac{\Delta(\lambda)}{\Delta^0(\lambda)} = 1, \quad \lim_{\lambda \to -\infty} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda_n - \lambda_n^0}{\lambda - \lambda_n^0}\right) = 1.$$

This yields

$$A = A^0 \prod_{n=0}^{\infty} \frac{\lambda_n}{\lambda_n^0},$$

and we arrive at (6). Relation (7) is proved similarly.

3. The Weyl sequence

Denote $d(\lambda) := S(T, \lambda)$. Zeros $\mathcal{V} = \{v_n\}_{n \geq 1}$ of $d(\lambda)$ (counting with multiplicities) coincide with the eigenvalues of the boundary value problem L_0 for equation (1) with Dirichlet boundary conditions $y(0) = y(T) = 0$. The function $d(\lambda)$ is called the characteristic function for L_0. Let $G_\delta := \{\rho : |\rho - (n\pi)/T| \geq \delta \ \forall n\}$. Then (see [1], [3])

$$\sqrt{v_n} = \frac{n\pi}{T} + \frac{q_1}{n\pi} + o\left(\frac{1}{n}\right), \quad n \to \infty, \tag{8}$$

$$d(\lambda) = \frac{T^3}{\pi^2} \prod_{n=1}^{\infty} \frac{v_n - \lambda}{n^2}, \tag{9}$$

$$|d(\lambda)| \geq C|\rho|^{-1} e^{|\tau|T}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^*. \tag{10}$$

Let $\Phi(x, \lambda)$ be the solution of equation (1) under the conditions $\Phi(0, \lambda) = 1$, $\Phi(T, \lambda) = 0$. Denote $M(\lambda) := \Phi'(0, \lambda)$. The function $M(\lambda)$ is called the Weyl function for L_0. Clearly,

$$M(\lambda) = -\frac{d_1(\lambda)}{d(\lambda)}, \tag{11}$$

$$\Phi(x, \lambda) = C(x, \lambda) + M(\lambda)S(x, \lambda), \tag{12}$$

where $d_1(\lambda) := C(T, \lambda)$. Relation (12) is used for solving inverse spectral problems for the boundary value problem L_0 by the method of spectral mappings [4, 14, 15]. Since $d_1(\lambda) = O(e^{|\tau|T})$, it follows from (10)-(11) that

$$|M(\lambda)| \leq C|\rho|, \quad \rho \in G_\delta, |\rho| > \rho^*. \tag{13}$$

For $|\rho| \to \infty$, one has

$$d_1(\lambda) = \cos\rho T + q_1 \frac{\sin\rho T}{\rho} + o\Big(\frac{e^{|\tau|T}}{\rho}\Big), \quad d(\lambda) = \frac{\sin\rho T}{\rho} - q_1 \frac{\cos\rho T}{\rho^2} + o\Big(\frac{e^{|\tau|T}}{\rho}\Big),$$

$$\dot{d}(\lambda) = \frac{1}{2\rho}\Big(\frac{T\cos\rho T}{\rho} + \frac{(q_1 T - 1)\sin\rho T}{\rho} + o\Big(\frac{e^{|\tau|T}}{\rho}\Big)\Big), \quad \dot{d}(\lambda) := \frac{d}{d\lambda} d(\lambda).$$

In particular, this yields

$$M(\lambda) = i\rho + o(1), \quad |\rho| \to \infty, \ \arg\rho \in [\delta, \pi - \delta], \tag{14}$$

Let m_n be the multiplicity of ν_n ($\nu_n = \nu_{n+1} = \ldots = \nu_{n+m_n-1}$). Denote

$$S := \{n \ge 2 : \nu_{n-1} \ne \nu_n\} \cup \{1\}, \quad I := \{n \in S : m_n > 1\}.$$

It follows from (8) that I is a finite set, i.e. $m_n = 1$ for sufficiently large n ($n > n^*$). Fix $n \in S$. Using (11) we obtain that in a neighborhood of the point $\lambda = \nu_n$ the function $M(\lambda)$ has the representation

$$M(\lambda) = \sum_{\nu=0}^{m_n-1} \frac{M_{n+\nu}}{(\lambda - \nu_n)^{\nu+1}} + M_n^*(\lambda), \tag{15}$$

where $M_n^*(\lambda)$ is regular in $\lambda = \nu_n$, and the coefficients $M_{n+\nu}$, $\nu = \overline{0, m_n - 1}$ are calculated from $d_1^{(\nu)}(\nu_n)$ and $d^{(\nu+m_n)}(\nu_n)$ for $\nu = \overline{0, m_n - 1}$. More precisely,

$$M_{n+m_n-1-\nu} = -\frac{1}{d_{0n}^0}\Big(d_{\nu n}^1 + \sum_{k=0}^{\nu-1} M_{n+m_n-1-k} d_{\nu-k,n}^0\Big), \quad \nu = \overline{0, m_n - 1},$$

$$d_{\nu n}^1 := \frac{1}{\nu!} d_1^{(\nu)}(\nu_n), \quad d_{\nu n}^0 := \frac{1}{(\nu + m_n)!} d^{(\nu+m_n)}(\nu_n), \quad \nu = \overline{0, m_n - 1}.$$

In particular, if $m_n = 1$ (i.e. $n \in S \setminus I$), then

$$M_n = -\frac{d_1(\nu_n)}{\dot{d}(\nu_n)}. \tag{16}$$

The sequence $\{M_n\}_{n \ge 1}$ is called the Weyl sequence, and the data $\mathcal{D} := \{\nu_n, M_n\}_{n \ge 1}$ are called the spectral data. One has

$$\dot{d}(\nu_n) = \frac{(-1)^n T}{2\nu_n}\Big(1 + o\Big(\frac{1}{n}\Big)\Big), \quad d_1(\nu_n) = (-1)^n\Big(1 + o\Big(\frac{1}{n}\Big)\Big), \quad n \to \infty. \tag{17}$$

Using (8), (16) and (17) we get

$$M_n = -\frac{2n^2\pi^2}{T^3}\Big(1 + o\Big(\frac{1}{n}\Big)\Big), \quad n \to \infty. \tag{18}$$

Take $\theta > 0$ such that $d(\pm i\theta) \ne 0$. For $n \in S$, we denote

$$b_{\nu,n} := \frac{1}{\nu!}\Big(\frac{z}{z^2 + \theta^2}\Big)^{(\nu)}_{|z=\nu_n}, \quad \nu = \overline{0, m_n - 1}.$$

Let us consider the function

$$N(\lambda) := \sum_{n \in S} \sum_{v=0}^{m_n-1} \left(\frac{1}{(\lambda + v_n)^{v+1}} + b_{v,n} \right) M_{n+v}. \tag{19}$$

By virtue of (8) and (18),

$$\left(\frac{1}{\lambda - v_n} + \frac{v_n}{v_n^2 + \theta^2} \right) M_n = \left(\frac{\theta^2 + \lambda v_n}{(\lambda - v_n)(v_n^2 + \theta^2)} \right) M_n = O\left(\frac{1}{n^2} \right), \quad n \to \infty,$$

and consequently, the series in (19) converges absolutely and uniformly in λ on compacts without $\{v_n\}$.

Theorem 2. *The specification of the spectral data $\mathscr{D}$ uniquely determines the Weyl function by the formula*

$$M(\lambda) = N(\lambda) + a, \tag{20}$$

where $N(\lambda)$ is defined by (19)*, and*

$$a = \lim_{\lambda \to -\infty} (-N(\lambda) + i\rho). \tag{21}$$

Proof. Consider the functions

$$J_N(\lambda) := \frac{1}{2\pi i} \int_{\Gamma_N} \left(\frac{1}{\lambda - z} + \frac{z}{z^2 + \theta^2} \right) M(z) \, dz, \tag{22}$$

where $\Gamma_N = \{z : |z| = \frac{\pi^2}{T^2}(N + 1/2)^2\}$. By virtue of (13),

$$\left(\frac{1}{\lambda - z} + \frac{z}{z^2 + \theta^2} \right) M(z) = O\left(\frac{1}{z^{3/2}} \right).$$

Therefore, the integral in (22) converges absolutely and uniformly in λ on compacts without Γ_N, and

$$\lim_{N \to \infty} J_N(\lambda) = 0. \tag{23}$$

On the other hand one can calculate $J_N(\lambda)$, using the residue theorem. This yelds

$$J_N(\lambda) = -M(\lambda) + \sum_{n \in S, \, v_n \in int\Gamma_N} \operatorname*{Res}_{z=v_n} \left(\frac{1}{\lambda - z} + \frac{z}{z^2 + \theta^2} \right) M(z) + a,$$

$$a = \operatorname*{Res}_{z=i\theta} \frac{z}{z^2 + \theta^2} M(z) + \operatorname*{Res}_{z=-i\theta} \frac{z}{z^2 + \theta^2} M(z).$$

By virtue of (15), one has

$$\operatorname*{Res}_{z=v_n} \frac{M(z)}{\lambda - z} = \sum_{v=0}^{m_n-1} \frac{M_{n+v}}{(\lambda - v_n)^{v+1}}, \quad \operatorname*{Res}_{z=v_n} \frac{zM(z)}{z^2 + \theta^2} = \sum_{v=0}^{m_n-1} b_{v,n} M_{n+v}.$$

Taking (23) and (14) into account we arrive at (20) and (21).

4. Inverse problems

In this section we formulate and study an inverse problem of recovering L from its spectral data. The main result is Theorem 3, where an algorithm for the solution of the inverse problem is provided, and the uniqueness of the solution is proved. Denote

$$D(\lambda) = \alpha\varphi(T,\lambda) + \beta S'(T,\lambda), \quad Q(\lambda) = \alpha\varphi(T,\lambda) - \beta S'(T,\lambda).$$

Then

$$\varphi(T,\lambda) = \frac{1}{2\alpha}\Big(D(\lambda) + Q(\lambda)\Big), \quad S'(T,\lambda) = \frac{1}{2\beta}\Big(D(\lambda) - Q(\lambda)\Big).$$

Since $\varphi(x,\lambda)S'(x,\lambda) - \varphi'(x,\lambda)S(x,\lambda) \equiv 1$, it follows that

$$Q^2(\lambda) = D^2(\lambda) - 4\alpha\beta(1 + \varphi'(T,\lambda)S(T,\lambda)), \tag{24}$$

and consequently,

$$\dot{Q}(\lambda)Q(\lambda) = \dot{D}(\lambda)D(\lambda) - 2\alpha\beta(\dot{\varphi}'(T,\lambda)S(T,\lambda) + \varphi'(T,\lambda)\dot{S}(T,\lambda)). \tag{25}$$

Let $n \in S$. Denote

$$\omega_n = \begin{cases} 0, & Q(\nu_n) = 0, \\ +1, & Q(\nu_n) \neq 0, \ \arg Q(\nu_n) \in [0,\pi), \\ -1, & Q(\nu_n) \neq 0, \ \arg Q(\nu_n) \in [\pi,2\pi), \end{cases}$$

$\omega_{n\nu} := d_1^{(\nu)}(\nu_n)$, $\nu = \overline{0, m_n - 1}$, $I_0 = \{n \in I : \ \omega_n = 0\}$, $I_1 = \{n \in I : \ \omega_n \neq 0\}$. The sequence $\Omega = \{\omega_n\}_{n \in S} \cup \{\omega_{n\nu}\}_{n \in I_0, \ \nu = \overline{1, m_n - 1}}$ is called the Ω- sequence for L. We note that if $I = \emptyset$ (i.e. $m_n = 1$ for all n), then $\Omega = \{\omega_n\}_{n \geq 1}$. For example, $I = \emptyset$, if $q(x), h, \alpha, \beta$ are real, i.e. in the self-adjoint case. Let α and β be known a priori and fixed. The inverse problem is formulated as follows.

Inverse problem 1. Given $\Lambda, \mathcal{V}$ and Ω, construct $q(x), h$ and T.

Obviously, in general it is not possible to recover uniquely the coefficients α and β, but it is possible to calculate γ. Note that Inverse problem 1 is a generalization of the classical inverse problems in the selfadjoint case (see [5]-[13]).

The solution of Inverse problem 1 can be constructed as follows. Using (8) we calculate T. By virtue of (6)-(7) and (9) we find $d(\lambda)$ and $\Delta(\lambda)$. Then we calculate $D(\lambda) = \Delta(\lambda) + (1 + \alpha\beta)$. It follows from (24) that $Q^2(\nu_n) = D^2(\nu_n) - 4\alpha\beta$, $n \in S$, and consequently,

$$Q(\nu_n) = \omega_n\sqrt{D^2(\nu_n) - 4\alpha\beta}, \quad n \in S. \tag{26}$$

Since $\varphi(T, v_n) = C(T, v_n)$, we construct

$$\omega_{n0} = \frac{1}{2\alpha}\Big(D(v_n) + Q(v_n)\Big), \quad n \in S. \tag{27}$$

Now we are going to calculate the Weyl sequence $\{M_n\}_{n \geq 1}$.

Case 1. Let $n \in S$, $n \notin I$ (i.e. $m_n = 1$). Taking (16) into account we get

$$M_n = \frac{\omega_{n0}}{\dot{d}(v_n)}. \tag{28}$$

Case 2. Let $n \in I_1$ (i.e. $m_n > 1$, $\omega_n \neq 0$). Then it follows from (25) that

$$(\dot{Q}(\lambda)Q(\lambda))^{(v-1)}_{|\lambda=v_n} = (\dot{D}(\lambda)D(\lambda))^{(v-1)}_{|\lambda=v_n}, \quad v = \overline{1, m_n - 1}. \tag{29}$$

Using (29) we find $Q^{(v)}(v_n)$, $v = \overline{1, m_n - 1}$. Since $d_1(\lambda) = \frac{1}{2\alpha}\Big(D(\lambda) + Q(\lambda)\Big) - hd(\lambda)$, we construct ω_{nv}, $v = \overline{1, m_n - 1}$ by the formula

$$\omega_{nv} = \frac{1}{2\alpha}\Big(D^{(v)}(v_n) + Q^{(v)}(v_n)\Big), \quad v = \overline{1, m_n - 1}. \tag{30}$$

Case 3. Let $n \in I_0$ (i.e. $m_n > 1$, $\omega_n = 0$). Then ω_{nv}, $v = \overline{1, m_n - 1}$ are given a priori.

Thus, we have constructed the Weyl sequence $\{M_n\}_{n \geq 1}$. It is also possible to construct the Weyl function $M(\lambda)$ by (20)-(21). Our calculations show that Inverse problem 1 is reduced to the following inverse problem.

Inverse problem 2. Given $\mathscr{D} = \{v_n, M_n\}_{n \geq 1}$, construct $q(x)$.

This non-selfadjoint inverse problem was solved in [15]. For this inverse problem the uniqueness theorem was proved and a constructive procedure for its solution was provided in [15]. Moreover, necessary and sufficient conditions for the solvability of Inverse problem 2 were also established. We note that for the *selfadjoint* case Inverse problem 2 is equivalent to the classical inverse Sturm-Liouville problem (see [1, 2, 3, 4]).

Thus, we have obtained a procedure for the solution on Inverse problem 1 and proved its uniqueness, i.e. the following theorem holds.

Theorem 3. *The specification of* Λ, $\mathcal{V}$ *and* Ω *uniquely determines* $q(x)$, h *and* T. *The solution of Inverse problem* 1 *can be found by the following algorithm.*

Algorithm 1. *Given* Λ, $\mathcal{V}$ *and* Ω.
(1) *Find T, using* (8).
(2) *Construct* $\Delta(\lambda)$ *and* $d(\lambda)$ *by* (6)-(7) *and* (9).
(3) *Calculate* $D(\lambda) = \Delta(\lambda) + (1 + \alpha\beta)$.

(4) *Find $Q(v_n)$, $n \in S$, by* (26).

(5) *Calculate ω_{n0}, $n \in S$, by* (27).

(6) *Construct $Q^{(v)}(v_n)$, $n \in I_1$, $v = \overline{1, m_n - 1}$ using* (29).

(7) *Find ω_{nv}, $n \in I_1$, $v = \overline{1, m_n - 1}$, via* (30).

(8) *Calculate the Weyl sequence $\{M_n\}_{n \geq 1}$ using* (28) *and the recurrent formula*

$$M_{n+m_n-1-v} = -\frac{1}{d_{0n}^0}\left(d_{vn}^1 + \sum_{k=0}^{v-1} M_{n+m_n-1-k}d_{v-k,n}^0\right), \quad n \in S, \, v = \overline{0, m_n - 1}, \, d_{vn}^1 := \frac{1}{v!}\omega_{nv}.$$

(9) *Find the potential $q(x)$ by solving Inverse problem 2 (see [15]).*

(10) *Calculate the coefficient h, using* (3).

5. Periodic boundary conditions

In this section we study the periodic boundary value problem which is a particular case of L. For convenience of readers, we formulate here the results from [8] (published in 1981 only in Russian) where necessary and sufficient conditions for the solvability of the inverse problem were obtained for the selfadjoint periodic case. In [5] one can find another conditions for the characterization of the periodic spectrum.

Let us consider the periodic boundary value problem $L' := L(q(x), 0, 1, 1, \pi)$ with $\alpha = \beta = 1$, $h = 0$, $T = \pi$, and real-valued potential $q(x) \in L_2(0, \pi)$. Denote

$$p(\lambda) = 1 - \frac{1}{2}\Big(C(\pi, \lambda) + S'(\pi, \lambda)\Big).$$

The zeros $\Lambda = \{\lambda_n\}_{n \geq 0}$ of $p(\lambda)$ coincide with the eigenvalues of L'. It is known (see [1]) that λ_n are real, and

$$\lambda_{2n-1} = (2n)^2 + a + \kappa'_{2n-1}, \, \lambda_{2n} = (2n)^2 + a + \kappa'_{2n}, \, \{\kappa'_n\} \in l_2, \, \lambda_n \leq \lambda_{n+1}, \, \lambda_n < \lambda_{n+2}, \tag{31}$$

where $a = \frac{1}{\pi}\int_0^\pi q(t)dt$. Moreover, it follows form (7) that

$$p(\lambda) = \frac{\pi^2}{2}(\lambda - \lambda_0)\prod_{n=1}^\infty \frac{\lambda_n - \lambda}{\lambda_n^0}, \quad \lambda_{2n-1}^0 = \lambda_{2n}^0 = (2n)^2. \tag{32}$$

Denote $s(\lambda) = p(\lambda) - 2$. The zeros $\{\mu_n\}_{n \geq 1}$ of $s(\lambda)$ coincide with the eigenvalues of the anti-periodic boundary value problem $L'' := L(q(x), 0, -1, -1, \pi)$. It is known that μ_n are real, $\lambda_0 < \mu_1 \leq \mu_2 < \lambda_1 \leq \lambda_2 < \ldots$, and

$$\mu_{2n-1} = (2n-1)^2 + a + \kappa''_{2n-1}, \quad \mu_{2n} = (2n-1)^2 + a + \kappa''_{2n}, \quad \{\kappa''_n\} \in l_2.$$

Moreover,

$$\max_{\lambda \in [\lambda_{2n}, \lambda_{2n+1}]} p(\lambda) \geq 2. \tag{33}$$

We will show that condition (33) is also sufficient for the solvability of the inverse problem. It is known (see [1, 2, 3, 4]) that the zeros $\{\nu_n\}_{n\geq 1}$ of $d(\lambda) := S(\pi, \lambda)$ are real, and

$$\nu_n = n^2 + a + \kappa_n^0, \quad \{\kappa_n^0\} \in l_2, \quad \nu_{2n} \in [\lambda_{2n-1}, \lambda_{2n}], \quad \nu_{2n-1} \in [\mu_{2n-1}, \mu_{2n}]. \tag{34}$$

Denote $Q(\lambda) = C(\pi, \lambda) - S'(\pi, \lambda)$, $\Omega' = \{\omega_n\}_{n\geq 1}$, where $\omega_n = \text{sign}\, Q(\nu_n)$, i.e. $\omega_n = 0$, if $Q(\nu_n) = 0$, and $\omega_n = \pm 1$, if $\pm Q(\nu_n) > 0$. The sequence Ω' is called the sign sequence for q.

Now we will give the characterization of the spectrum of L'. We will say that a real sequence $\{\lambda_n\}_{n\geq 0} \in \mathscr{S}$ if there exists L' such that $\{\lambda_n\}_{n\geq 0}$ is the spectrum of L'. Denote by J the set of sequences $\{\omega_n\}_{n\geq 1}$ such that $\omega_n = 0$ if $p(\nu_n)s(\nu_n) = 0$, and $\omega_n = \pm 1$ if $p(\nu_n)s(\nu_n) \neq 0$.

Theorem 4. *Let the real numbers $\{\lambda_n\}_{n\geq 0}$ of the form* (31) *be given. For $\{\lambda_n\}_{n\geq 0} \in \mathscr{S}$ it is necessary and sufficient that* (33) *holds, where $p(\lambda)$ is constructed by* (32). *Moreover, if we additionally have real numbers $\{\nu_n\}_{n\geq 1}$ satisfying* (34), *where $\{\mu_n\}_{n\geq 1}$ are zeros of $s(\lambda) = p(\lambda) - 2$, and the sequence $\{\omega_n\}_{n\geq 1} \in J$, then there exists a unique real function $q(x) \in L_2(0, \pi)$ for which $\{\lambda_n\}_{n\geq 0}$ and $\{\nu_n\}_{n\geq 1}$ are the eigenvalues of L' and L_0, respectively, and $\{\omega_n\}_{n\geq 1}$ is the sign sequence for q.*

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