TAMKANG JOURNAL OF MATHEMATICS
Volume 44, Number 3, 271-278, Autumn 2013
doi:10.5556/j.tkjm.44.2013.1108
Available online at http://journals.math.tku.edu.tw/

A NEW SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE AND FIXED SECOND COEFFICIENTS

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Abstract. In this paper we introduce and study a subclass $\mathscr{M}_P(\alpha,\lambda,c)$ of meromorphic univalent functions. We obtain coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and meromorphically convexity for the class $\mathscr{M}_P(\alpha,\lambda,c)$ by fixing the second coefficient. Further, it is shown that the class $\mathscr{M}_P(\alpha,\lambda,c)$ is closed under convex linear combination.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $\Sigma_{\mathscr{S}}$, $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$, $(0 \le \alpha < 1)$ denote the subclasses of Σ that are meromorphically univalent functions , meromorphically starlike functions of order α and meromophically convex functions of order α respectively. Analytically, $f \in \Sigma^*(\alpha)$ if and only if, f is of the form (1.1) and satisfies

$$-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{U},$$

similarly, $f \in \Sigma_K(\alpha)$, if and only if, f is of the form (1.1) and satisfies

$$-\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{U},$$

Corresponding author: S. Sivasubramanian.
2010 *Mathematics Subject Classification*. 30C80, 30C45.
Key words and phrases. Meromorphic functions, meromorphically starlike functions, meromorphically convex functions..

and similar other classes of meromorphically univalent functions have been extensively studied by Altintas et al. [1], Aouf [2, 3], Ganigi and Uralegaddi [6], Kulkarni and Joshi [9], Mogra et al. [11], Uralegadi [12], Uralegaddi and Ganigi [13] and Uralegaddi and Somanatha [14] and others.

Let Σ_P be the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n, \; a_n \geq 0, \tag{1.2}$$

that are analytic and univalent in $\mathbb{U}^*$. For functions $f \in \Sigma$ given by (1.1) and $g \in \Sigma$ given by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n, \tag{1.3}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.4}$$

Now, in the following definition, we define a subclass $\mathcal{M}_P(\alpha,\lambda)$ for functions in the class Σ_P.

Definition 1.1. For $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$, let $\mathcal{M}(\alpha,\lambda)$ denote a subclass of Σ consisting of functions of the form (1.1) satisfying the condition that

$$\Re\left(\frac{zf'(z)}{(\lambda-1)f(z)+\lambda z f'(z)}\right) > \alpha, \;\; z \in \mathbb{U}^*. \tag{1.5}$$

Furthermore, we say that a function $f \in \mathcal{M}_P(\alpha,\lambda)$, whenever $f(z)$ is of the form (1.2). For the class $\mathcal{M}_P(\alpha,\lambda)$, the following characterization was given by Kavitha et al., [8].

Theorem 1.1. *Let $f \in \Sigma_P$ be given by (1.2). Then $f \in \mathcal{M}_P(\alpha,\lambda)$ if and only if*

$$\sum_{n=1}^{\infty} [n+\alpha-\alpha\lambda(1+n)]\; a_n \leq (1-\alpha). \tag{1.6}$$

For a function defined by (1.2) and in the class $\mathcal{M}_P(\alpha,\lambda)$, Theorem 1.1, immediately yields

$$a_1 \leq \frac{(1-\alpha)}{1+\alpha(1-2\lambda)}. \tag{1.7}$$

Hence we may take

$$a_1 = \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}, \;\; c\,(0<c<1). \tag{1.8}$$

Motivated by the works of Aouf and Darwish [4], Aouf and Joshi [5], Ghanim and Darus [7], Magesh et al. [10] and Uralegaddi [12], we now introduce the following class of functions and use the similar techniques to prove our results.

Let $\mathcal{M}_P(\alpha,\lambda,c)$ be the subclass of $\mathcal{M}_P(\alpha,\lambda)$ consisting of functions of the form

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}z+\sum_{n=2}^{\infty}[n+\alpha-\alpha\lambda(1+n)]a_n z^n \tag{1.9}$$

where $0<c<1$.

The main object of this paper is to obtain coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and meromorphically convexity for the class $\mathcal{M}_P(\alpha,\lambda,c)$ by fixing the second coefficient. Further, it is shown that the class $\mathcal{M}_P(\alpha,\lambda,c)$ is closed under convex linear combination.

2. Main results

In our first theorem, we now find out the coefficient inequality for the class $\mathcal{M}_P(\alpha,\lambda,c)$.

Theorem 2.1. *Let the function $f(z)$ defined by (1.9). Then $f\in\mathcal{M}_P(\alpha,\lambda,c)$ if and only if,*

$$\sum_{n=2}^{\infty}[n+\alpha-\alpha\lambda(1+n)]a_n\le(1-\alpha)(1-c). \tag{2.1}$$

The result is sharp.

Proof. By putting

$$a_1=\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)},\quad 0<c<1, \tag{2.2}$$

in (1.6), the result is easily derived. The result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}z+\frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]}z^n,\quad n\ge 2. \tag{2.3}$$

□

Corollary 2.2. *If the function f defined by (1.9) is in the class $\mathcal{M}_P(\alpha,\lambda,c)$, then*

$$a_n\le\frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]},\quad n\ge 2. \tag{2.4}$$

The result is sharp for the function $f(z)$ given by (2.3).

Next we obtain growth and distortion properties for the class $\mathcal{M}_P(\alpha,\lambda,c)$.

Theorem 2.3. *If the function $f(z)$ defined by (1.9) is in the class $\mathcal{M}_P(\alpha,\lambda,c)$ for $0<|z|=r<1$, then we have*

$$\frac{1}{r}-\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}r-\frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)}r^2\le|f(z)|$$

$$\leq \frac{1}{r} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} r + \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r^2.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z + \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} z^2.$$

Proof. Since $f \in \mathcal{M}_P(\alpha, \lambda, c)$, Theorem 2.1 yields,

$$a_n \leq \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]}, \quad n \geq 2. \tag{2.5}$$

Thus, for $0 < |z| = r < 1$

$$|f(z)| \leq \frac{1}{|z|} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} |z| + \sum_{n=2}^{\infty} a_n |z|^n$$

$$\leq \frac{1}{r} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} r + r^2 \sum_{n=2}^{\infty} a_n \leq \frac{1}{r} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} r + \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r^2$$

and

$$|f(z)| \geq \frac{1}{|z|} - \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} |z| - \sum_{n=2}^{\infty} a_n |z|^n$$

$$\geq \frac{1}{r} - \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} r - r^2 \sum_{n=2}^{\infty} a_n \geq \frac{1}{r} - \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} r - \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r^2.$$

Thus the proof of the theorem is complete. □

Theorem 2.4. *If the function $f(z)$ defined by (1.9) is in the class $\mathcal{M}_P(\alpha, \lambda, c)$ for $0 < |z| = r < 1$, then we have*

$$\frac{1}{r^2} - \frac{(1-\alpha)c}{1+(1-\alpha)} - \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r \leq |f'(z)|$$

$$\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} + \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r.$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z + \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} z^2.$$

Proof. In view of Theorem 2.1, it follows that

$$n a_n \leq \frac{2n(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]}, \quad n \geq 2. \tag{2.6}$$

Thus, for $0 < |z| = r < 1$ and making use of (2.6), we obtain

$$|f'(z)| \leq \left| \frac{-1}{z^2} \right| + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} + \sum_{n=2}^{\infty} n a_n |z|^{n-1}, \quad |z| = r$$

$$\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{1+(1-\alpha)} + r\sum_{n=2}^{\infty} na_n$$
$$\leq \frac{1}{r^2} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} + \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r$$

and

$$|f'(z)| \geq \left|\frac{-1}{z^2}\right| - \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} - \sum_{n=2}^{\infty} na_n|z|^{n-1}, \quad |z| = r$$
$$\geq \frac{1}{r^2} - \frac{(1-\alpha)c}{1+(1-\alpha)} - r\sum_{n=2}^{\infty} na_n$$
$$\geq \frac{1}{r^2} - \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} - \frac{(1-\alpha)(1-c)}{2+\alpha(1-3\lambda)} r.$$

Hence the result follows. □

Next, we shall show that the class $\mathcal{M}_P(\alpha,\lambda,c)$ is closed under convex linear combination.

Theorem 2.5. *If*

$$f_1(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z \tag{2.7}$$

and

$$f_n(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z + \sum_{n=2}^{\infty} \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]} z^n, \quad n \geq 2. \tag{2.8}$$

Then $f \in \mathcal{M}_P(\alpha,\lambda,c)$ if and only if it can expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z) \tag{2.9}$$

where $\mu_n \geq 0$ and $\sum_{n=2}^{\infty} \mu_n \leq 1$.

Proof. From (2.7)(2.8)(2.9), we have

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z + \sum_{n=2}^{\infty} \frac{(1-\alpha)(1-c)\mu_n}{[n+\alpha-\alpha\lambda(1+n)]} z^n.$$

Since

$$\sum_{n=2}^{\infty} \frac{(1-\alpha)(1-c)\mu_n}{[n+\alpha-\alpha\lambda(1+n)]} \frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)(1-c)}$$
$$= \sum_{n=2}^{\infty} \mu_n = 1-\mu_1 \leq 1$$

it follows from Theorem 1.1 that the function $f \in \mathcal{M}_P(\alpha,\lambda,c)$. Conversely, suppose that $f \in \mathcal{M}_P(\alpha,\lambda,c)$. Since

$$a_n \leq \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]}, \quad n \geq 2.$$

Setting

$$\mu_n = \frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)(1-c)} a_n$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

It follows that

$$f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z).$$

Hence the proof complete. □

Theorem 2.6. *The class $\mathcal{M}_P(\alpha,\lambda,c)$ is closed under linear combination.*

Proof. Suppose that the function f be given by (1.9), and let the function g be given by

$$g(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z + \sum_{n=2}^{\infty} |b_n| z^n, \quad n \geq 2.$$

Assuming that f and g are in the class $\mathcal{M}_P(\alpha,\lambda,c)$, it is enough to prove that the function H defined by

$$h(z) = \mu f(z) + (1-\mu) g(z), \quad 0 \leq \lambda \leq 1$$

is also in the class $\mathcal{M}_P(\alpha,\lambda,c)$. Since

$$h(z) = \frac{1}{z} + \frac{(1-\alpha)c}{1+\alpha(1-2\lambda)} z + \sum_{n=2}^{\infty} |a_n\mu + (1-\mu) b_n| z^n,$$

we observe that

$$\sum_{n=2}^{\infty} [n+\alpha-\alpha\lambda(1+n)] |a_n\mu + (1-\mu) b_n| \leq (1-\alpha)(1-c),$$

with the aid of Theorem 2.1. Thus $h \in \mathcal{M}_P(\alpha,\lambda,c)$. □

Next we determine the radii of meromophically starlikeness of order δ and meromophically convexity of order δ for functions in the class $\mathcal{M}_P(\alpha,\lambda,c)$.

Theorem 2.7. *Let the function $f(z)$ defined by (1.9) be in the class $\mathcal{M}_P(\alpha,\lambda,c)$, then we have*

(i) *f is meromophically starlike of order $\delta (0 \leq \delta < 1)$ in the disk $|z| < r_1(\alpha,\lambda,c,\delta)$ where $r_1(\alpha,\lambda,c,\delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\alpha)c}{1+\alpha(1-2\lambda)} r^2 + \frac{(n+2-\delta)(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]} r^{n+1} \leq (1-\delta), \; n \geq 2.$$

(ii) *f is meromophically convex of order $\delta (0 \leq \delta < 1)$ in the disk $|z| < r_2(\alpha,\lambda,c,\delta)$ where $r_2(\alpha,\lambda,c,\delta)$ is the largest value for which*

$$\frac{(3-\delta)(1-\alpha)c}{1+\alpha(1-2\lambda)} r^2 + \frac{n(n+2-\delta)(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]} r^{n+1} \leq (1-\delta), \; n \geq 2.$$

Each of these results is sharp for the function $f_n(z)$ given by (2.3).

Proof. It is enough to highlight that

$$\left|\frac{zf^{'}(z)}{f(z)}+1\right| \le 1-\delta, \ \ |z|<r_1.$$

Thus, we have

$$\left|\frac{zf^{'}(z)}{f(z)}+1\right| = \left|\frac{\frac{-1}{z}+\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}z+\sum\limits_{n=2}^{\infty} na_nz^n+\frac{1}{z}+\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}z+\sum\limits_{n=2}^{\infty}a_nz^n}{\frac{1}{z}+\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}z+\sum\limits_{n=2}^{\infty}a_nz^n}\right|. \tag{2.10}$$

Hence (2.10) holds true if

$$\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}r^2+\sum_{n=2}^{\infty}(n+1)a_nr^{n+1} \le (1-\delta)\left[1-\frac{(1-\alpha)c}{1+\alpha(1-2\lambda)}r^2-\sum_{n=2}^{\infty}a_nr^{n+1}\right], \tag{2.11}$$

or,

$$\frac{(3-\delta)(1-\alpha)c}{1+\alpha(1-2\lambda)}r^2+\sum_{n=2}^{\infty}(n+2-\delta)a_nr^{n+1} \le (1-\delta) \tag{2.12}$$

and it follows that from (2.1), we may take

$$a_n \le \frac{(1-\alpha)(1-c)}{[n+\alpha-\alpha\lambda(1+n)]}\mu_n, \ \ n\ge 2, \tag{2.13}$$

where $\mu_n \ge 0$ and $\sum\limits_{n=2}^{\infty}\mu_n \le 1$.

For each fixed r, we choose the positive integer $n_0 = n_0(r_0)$ for which

$$\frac{(n+2-\delta)}{[n+\alpha-\alpha\lambda(1+n)]}r^{n+1},$$

is maximal. Then it follows that

$$\sum_{n=2}^{\infty}(n+2-\delta)a_nr^{n+1} \le \frac{(n_0+2-\delta)(1-\alpha)(1-c)}{[n_0+\alpha-\alpha\lambda(1+n_0)]}r^{n_0+1}. \tag{2.14}$$

Then f is starlike of order δ in $0<|z|<r_1(\alpha,\lambda,c,\delta)$ provided that

$$\frac{(3-\delta)(1-\alpha)c}{1+\alpha(1-2\lambda)}r^2+\frac{(n_0+2-\delta)(1-\alpha)(1-c)}{[n_0+\alpha-\alpha\lambda(1+n_0)]}r^{n_0+1} \le (1-\delta). \tag{2.15}$$

We find the value $r_0 = r_0(k,c,\delta,n)$ and the corresponding integer $n_0(r_0)$ so that

$$\frac{(3-\delta)(1-\alpha)c}{1+\alpha(1-2\lambda)}r_0^2+\frac{(n_0+2-\delta)(1-\alpha)(1-c)}{[n_0+\alpha-\alpha\lambda(1+n_0)]}r_0^{n_0+1} = (1-\delta). \tag{2.16}$$

It is the value for which the function $f(z)$ is starlike in $0<|z|<r_0$.

(ii) In a similar manner, we can prove our result providing the radius of meromorphically convexity of order δ $(0\le\delta<1)$ for functions in the class $\mathcal{M}_P(\alpha,\lambda,c)$, so we skip the proof of (ii). □

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