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ESTIMATION OF DERIVATIVES FOR REGULAR POSITIVE REAL PART FUNCTIONS

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Abstract. In this paper, we mainly discuss the problem of estimating the nth derivative of regular positive real part functions: $g(z) = c_0 + c_1z + \cdots + c_nz^n + \cdots$, which is regular in $|z| < 1$ and $\operatorname{Re} g(z) > 0$. With the principle of inductive method and the characters of regular positive real part functions, the estimation of the nth derivative for the function $g(z)$ is presented. The derivative estimation for positive functions with real part has been solved completely.

1. Introduction

Consider the following family of the function:

$$B = \{\varphi(z) \mid \varphi(z) = c + c_1z + \cdots + c_nz^n + \cdots, \text{ and } |\varphi(z)| < 1\}.$$

It is well known that

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$

Also Pan and Liao ([3]) has obtained the derivative estimation for the family B of the functions, i.e.. Let $\varphi(z) = c_0 + c_1z + \cdots + c_nz^n + \cdots$ be regular in $|z| < 1$ and $|\varphi(z) < 1$, then

$$|\varphi''(z)| \leq \frac{2(1 + |z|)}{(1 - |z|^2)^2}(1 - |\varphi(z)|^2).$$

And at the same time they gave the following estimation of first and second derivatives of regular function with positive real part:

$$|g'(z)| \leq \frac{2\operatorname{Re} g(z)}{1 - |z|^2}, \tag{1}$$

$$|g''(z)| = \frac{4(1 + |z|)}{(1 - |z|^2)^2}\operatorname{Re} g(z). \tag{2}$$

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We have to point out that there is an error in the Pan and Liao's ([3]) 3rd derivative estimation and Yuan ([5]) got the universal formula for (n^{th}) derivative of regularly bounded functions. This paper is to investigate the derivative estimation for regular function with positive real part further to obtain its estimation, and consequently the special estimation for derivative is generalized to usual one.

2. Main Result

Theorem. *Let $g(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ be regular in $|z| < 1$ and $\operatorname{Re} g(z) > 0$, then we have*

$$|g^{(n)}(z)| \le \frac{2n! \sum_{m=0}^{n-1} I(n,m)|z|^m}{(1-|z|^2)^n} \operatorname{Re} g(z), \tag{3}$$

where

$$I(n,0) = 1,\ I(n,1) = n-1,\ I(n,m) = \sum_{k=1}^{m} \binom{n-1}{n-k-1} I(n-k, m-k),$$
$$m \le n-1,\ m = 1, 2, \ldots, n-1.$$

Some Lemma was given before we complete the proof of Theorem.

Lemma 1.([1]) *Let $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$ be regualr in $|z| < 1$ and $|\varphi(z)| < 1$, then we have*

$$\varphi\left(\frac{s+z}{1+\bar{z}s}\right) = \sum_{n=0}^{\infty} g_n(z) s^n,$$

where $|s| < 1$, $g_0 = \varphi(z)$,

$$g_n(z) = \sum_{j=0}^{n-1} \frac{(-1)^j \binom{n-1}{j}}{(n-j)!} \bar{z}^j (1-|z|^2)^{n-j} \varphi^{(n-j)}(z), \quad (n \ge 1). \tag{4}$$

Lemma 2. *Let $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$ be regualr in $|z| < 1$ and $|\varphi(z)| < 1$, then*

$$\left| \frac{\varphi^{(n)}(z)(1-|z|^2)^n}{n!} \right| \le (1-|\varphi(z)|^2) + \sum_{v=1}^{n-1} \left[\binom{n-1}{j-1} |z|^{n-v} (1-|z|^2)^v \frac{|\varphi^{(v)}(z)|}{v!} \right]. \tag{5}$$

Proof. From Lemma 1

$$\Phi(s) = \varphi\left(\frac{s+z}{1+\bar{z}s}\right) = \sum_{n=0}^{\infty} g_n(z) s^n,$$

and its general term is given as

$$g_n(z)=\sum_{j=0}^{n-1}\frac{(-1)^j\binom{n-1}{j}}{(n-j)!}\bar{z}^j(1-|z|^2)^{n-j}\varphi^{(n-j)}(z),\quad (n\geq 1).$$

Since coefficients relations of regular functions: $|c_n|\leq 1-|c_0|^2$, we have

$$|\Phi^{(n)}(0)|=n!|g_n(z)|\leq n!(1-|\varphi(z)|^2);$$

Combining $|g_n(z)|\leq 1-|\varphi(z)|^2$ and the above form yields

$$\begin{aligned}1-|\varphi(z)|^2&\geq\left|\sum_{j=0}^{n-1}\frac{(-1)^j\binom{n-1}{j}}{(n-j)!}\bar{z}^j(1-|z|^2)^{n-j}\varphi^{(n-j)}(z)\right|\\&\geq\left|\frac{(1-|z|^2)^n}{n!}\varphi^{(n)}(z)\right|-\left|\sum_{j=0}^{n-1}\frac{(-1)^j\binom{n-1}{j}}{(n-j)!}\bar{z}^j(1-|z|^2)^{n-j}\varphi^{(n-j)}(z)\right|,\end{aligned}$$

Hence

$$\left|\frac{(1-|z|^2)^n}{n!}\varphi^{(n)}(z)\right|\leq(1-|\varphi(z)|^2)+\sum_{j=0}^{n-1}\frac{\binom{n-1}{j}}{(n-j)!}|z|^j(1-|z|^2)^{n-j}|\varphi^{(n-j)}(z)|,$$

and let $v=n-j$ we complete the proof.

Lemma 3. *If $g(z)=c_0+c_1z+c_2z^2+\cdots+c_nz^n+\cdots$ be regualr in $|z|<1$ and $\operatorname{Re}g(z)>0$, then*

$$\left|\frac{g^{(n)}(z)(1-|z|^2)^n}{n!}\right|\leq 2\operatorname{Re}g(z)+\sum_{v=1}^{n-1}\left[\binom{n-1}{v-1}|z|^{n-v}(1-|z|^2)^v\frac{|g^{(v)}(z)|}{v!}\right].\qquad(6)$$

Proof. Consider the function

$$G(s)=g\left(\frac{z+s}{1+\bar{z}s}\right),\quad (|s|<1).$$

Note that coefficients of expansion for positively real part function

$$|a_n|\leq 2\operatorname{Re}\{a_0\},$$

we reach

$$|G^{(n)}(0)|\leq 2n!\operatorname{Re}g(z).$$

From Lemma 1, we get

$$G^{(n)}(0)=\sum_{j=0}^{n-1}\frac{(-1)^j\binom{n-1}{j}}{(n-j)!}\bar{z}^j(1-|z|^2)^{n-j}g^{(n-j)}(z),\quad (n\geq 1).$$

So
$$\left|\sum_{j=0}^{n-1}\frac{(-1)^j\binom{n-1}{j}}{(n-j)!}\bar{z}^j(1-|z|^2)^{n-j}g^{(n-j)}(z)\right|\le 2n!\,\mathrm{Re}\,g(z),\quad (n\ge 1).$$
By Lemma 2, triangle inequality and let $v=n-j$, then we obtain the following inequality
$$\left|\frac{g^{(n)}(z)(1-|z|^2)^n}{n!}\right|\le 2\mathrm{Re}\,g(z)+\sum_{v=1}^{n-1}\left[\binom{n-1}{v-1}|z|^{n-v}(1-|z|^2)^v\frac{|g^{(v)}(z)|}{v!}\right],$$
and this completes the proof.

Now let's prove the Theorem by the principle of inductive method.

Let $n=1$, we have $I(1,0)=1$, and hence
$$|g'(z)|\le\frac{2\mathrm{Re}\,g(z)}{1-|z|^2},$$
that is, the Theorem holds.

Let $n=2$ and we get $I(2,1)=\binom{1}{0}I(1,0)=1$, $I(2,0)=1$ and
$$|g''(z)|\le\frac{2\cdot 2!(1+|z|)}{(1-|z|^2)^2}\mathrm{Re}\,g(z),$$
i.e., the Theorem holds.

Suppose Theorem holds in case $n=k$, equivalently,
$$|g^{(k)}(z)|\le\frac{2\cdot k!\,\mathrm{Re}\,g(z)}{(1-|z|^2)^k}\sum_{m=0}^{k-1}I(k,m)|z|^m;\tag{7}$$
Next we prove the Theorem in case $n=k+1$. Begin with the n^{th} derivative of
$$G(s)=g\left(\frac{z+s}{1+\bar{z}s}\right),$$
and let $s=0$ and note Lemma 3, we arrive at
$$\left|\frac{g^{(k+1)}(z)(1-|z|^2)^{k+1}}{(k+1)!}\right|\le 2\mathrm{Re}\,g(z)+\sum_{v=1}^{k}\left[\binom{k}{v-1}|z|^{k-v+1}(1-|z|^2)^v\frac{|g^{(v)}(z)|}{v!}\right].\tag{8}$$
Now substituting (7) into (8) yields,
$$\left|\frac{g^{(k+1)}(z)(1-|z|^2)^{k+1}}{(k+1)!}\right|$$
$$\le 2\mathrm{Re}\,g(z)+\sum_{v=1}^{k}2\left[\binom{k}{v-1}|z|^{k-v+1}\sum_{m=0}^{k-1}I(k,m)|z|^m\right]\mathrm{Re}\,g(z)$$

$$= 2\mathrm{Re}\, g(z)\left\{1+\sum_{v=1}^{k}\binom{k}{v-1}|z|^{k-v+1}\left[I(k,0)+I(k,1)|z|+\cdots+I(k,k-1)|z|^{k-1}\right]\right\}$$

$$= 2\mathrm{Re}\, g(z)\left\{1+\binom{k}{k-1}|z|+\left[\binom{k}{k-1}I(k,1)+\binom{k}{k-1}I(k-1,k-2)\right]|z|^2+\cdots\right.$$

$$\left.+\left[\binom{k}{k-1}I(k,k-1)+\binom{k}{k-2}I(k-1,k-2)+\cdots+\binom{k}{1}I(2,1)+\binom{k}{0}I(1,0)\right]|z|^k\right\}$$

$$= 2\mathrm{Re}\, g(z)\left\{1+k|z|+\left[\sum_{v=1}^{2}\binom{k}{k-v}I(k-v+1,2-v)\right]|z|^2\right.$$

$$+\left[\sum_{v=1}^{3}\binom{k}{k-v}I(k-v+1,3-v)\right]|z|^3+\cdots$$

$$\left.+\left[\sum_{v=1}^{s}\binom{k}{k-v}I(k-v+1,s-v)\right]|z|^s+\cdots+\left[\sum_{v=1}^{k}\binom{k}{k-v}I(k-v+1,k-v)\right]|z|^k\right\}$$

$$= 2\mathrm{Re}\, g(z)\sum_{m=0}^{k}I(k,m)|z|^m, \text{ i.e.}$$

$$|g^{(k+1)}(z)| \le \frac{2(k+1)!\,\mathrm{Re}\, g(z)}{(1-|z|^2)^{k+1}}\sum_{m=0}^{k}I(k+1,m)|z|^m.$$

Thus the proof is completed and we point out that the derivative estimation for positive functions with real part has been solved completely.

For an extremal function in the function class we will present our future presentation.

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