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# ON SOME OSTROWSKI TYPE INEQUALITIES VIA MONTGOMERY IDENTITY AND TAYLOR'S FORMULA

A. AGLIĆ ALJINOVIĆ AND J. PEČARIĆ

**Abstract**. A new extension of the weighted Montgomery identity is given, by using Taylor's formula, and used to obtain some Ostrowski type inequalities and the estimations of the difference of two integral means.

## 1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$, and $f':[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds [5]

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b P(x,t)\,f'(t)\,dt$$

where $P(x,t)$ is the Peano kernel, defined by

$$P(x,t)=\begin{cases}\dfrac{t-a}{b-a}, & a\le t\le x,\\[2mm] \dfrac{t-b}{b-a}, & x<t\le b.\end{cases}$$

Now, let's suppose $w:[a,b]\to[0,\infty\rangle$ is some probability density function, i.e. integrable function satisfying $\int_a^b w(t)\,dt=1$, and $W(t)=\int_a^t w(x)\,dx$ for $t\in[a,b]$, $W(t)=0$ for $t<a$ and $W(t)=1$ for $t>b$. The following identity (given by Pečarić in [6]) is the weighted generalization of Montgomery identity

$$f(x)=\int_a^b w(t)\,f(t)\,dt+\int_a^b P_w(x,t)\,f'(t)\,dt \tag{1.1}$$

where the weighted Peano kernel is

$$P_w(x,t)=\begin{cases}W(t), & a\le t\le x,\\ W(t)-1, & x<t\le b.\end{cases} \tag{1.2}$$

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In this paper we will extend the weighted Montgomery identity (1.2) using the Taylor's formula (Section 2.), and obtain some new Ostrowski type inequalities (Section 3.), as well as some generalizations of the estimations of the difference of two weighted integral means (Section 4).

## 2. An Extension of Montgomery Identity via Taylor's Formula

**Theorem 1.** *Let $f : I \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $I \subset \mathbb{R}$ an open interval, $a, b \in I$, $a < b$. If $w : [a, b] \to [0, \infty\rangle$ is some probability density function. Then the following identity hold*

$$f(x) = \int_a^b w(t) f(t)\, dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}\, ds + \frac{1}{(n-1)!} \int_a^b T_{w,n}(x,s) f^{(n)}(s)\, ds \tag{2.1}$$

*where*

$$T_{w,n}(x,s) = \begin{cases} \int_a^s w(u)(u-s)^{n-1}\, du, & a \leq s \leq x, \\ -\int_s^b w(u)(u-s)^{n-1}\, du, & x < s \leq b. \end{cases}$$

**Proof.** If we apply Taylor's formula with $f'(t)$, $(n \geq 2)$ we have

$$f'(t) = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{i!} (t-x)^i + \int_x^t f^{(n)}(s) \frac{(t-s)^{n-2}}{(n-2)!} ds.$$

By putting these two formulae in the weighted Montgomery identity (1.1) we obtain

$$f(x) = \int_a^b w(t) f(t)\, dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{i!} \int_a^b P_w(x,t)(t-x)^i\, dt + \int_a^b P_w(x,t) \left( \int_x^t f^{(n)}(s) \frac{(t-s)^{n-2}}{(n-2)!} ds \right) dt.$$

Now,

$$\begin{aligned} \int_a^x (t-x)^i W(t)\, dt &= \int_a^x (t-x)^i \left( \int_a^t w(s)\, ds \right) dt \\ &= \int_a^x w(s) \left( \int_s^x (t-x)^i\, dt \right) ds \\ &= -\frac{1}{i+1} \int_a^x w(s)(s-x)^{i+1}\, ds \end{aligned}$$

and similarly

$$\begin{aligned}\int_x^b (t-x)^i (W(t)-1)\,dt &= -\int_x^b (t-x)^i \left(\int_t^b w(s)\,ds\right) dt \\ &= -\int_x^b w(s) \left(\int_x^s (t-x)^i\,dt\right) ds \\ &= -\frac{1}{i+1}\int_x^b w(s)(s-x)^{i+1}\,ds.\end{aligned}$$

Further, we have

$$\int_a^x W(t)\left(\int_x^t f^{(n)}(s)(t-s)^{n-2}\,ds\right) dt = -\int_a^x f^{(n)}(s)\left(\int_a^s W(t)(t-s)^{n-2}\,dt\right) ds$$

and

$$\begin{aligned}&\int_a^s W(t)(t-s)^{n-2}\,dt = \int_a^s \left(\int_a^t w(u)\,du\right)(t-s)^{n-2}\,dt \\ &= \int_a^s w(u)\left(\int_u^s (t-s)^{n-2}\,dt\right) du = -\int_a^s w(u)\frac{(u-s)^{n-1}}{n-1}du.\end{aligned}$$

Similarly

$$\begin{aligned}&\int_x^b (W(t)-1)\left(\int_x^t f^{(n)}(s)(t-s)^{n-2}\,ds\right) dt \\ &= -\int_x^b f^{(n)}(s)\left(\int_s^b (1-W(t))(t-s)^{n-2}\,dt\right) ds\end{aligned}$$

and

$$\begin{aligned}&\int_s^b (1-W(t))(t-s)^{n-2}\,dt = \int_s^b \left(\int_t^b w(u)\,du\right)(t-s)^{n-2}\,dt \\ &= \int_s^b w(u)\left(\int_s^u (t-s)^{n-2}\,dt\right) du = \int_s^b w(u)\frac{(u-s)^{n-1}}{n-1}du.\end{aligned}$$

So the reminder in the weighted Taylor formula is

$$\begin{aligned}&\frac{1}{(n-1)!}\Big[\int_a^x f^{(n)}(s)\left(\int_a^s w(u)(u-s)^{n-1}\,du\right) ds \\ &+\int_x^b f^{(n)}(s)\left(-\int_s^b w(u)(u-s)^{n-1}\,du\right) ds\Big].\end{aligned}$$

**Remark 1.** In the special case, if we take $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ the equality (2.1) reduces to

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt - \sum_{i=0}^{n-2} f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)} + \frac{1}{(n-1)!}\int_a^b T_n(x,s)\,f^{(n)}(s)\,ds \tag{2.2}$$

where

$$T_n(x,s) = \begin{cases} \dfrac{-1}{n(b-a)}(a-s)^n, & a \le s \le x, \\ \dfrac{-1}{n(b-a)}(b-s)^n, & x < s \le b. \end{cases}$$

## 3. The Ostrowski Type Inequalities

In this section we generalize the results from [3] and [4].

**Theorem 2.** *Suppose that all the assumptions of Theorem 1 hold. Additionally assume that $(p,q)$ is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$, and $\left|f^{(n)}\right|^p$ on R-integrable function for some $n \ge 2$. Then we have*

$$\left|f(x) - \int_a^b w(t)f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}\,ds\right| \le \frac{1}{(n-1)!}\left(\int_a^b |T_{w,n}(x,s)|^q\,ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p. \tag{3.1}$$

*The constant $\frac{1}{(n-1)!}\left(\int_a^b |T_{w,n}(x,s)|^q\,ds\right)^{\frac{1}{q}}$ is sharp for $1 < p \le \infty$ and the best possible for $p = 1$.*

**Proof.** We use the identity (2.1) and apply the Hölder inequality to obtain

$$\left|f(x) - \int_a^b w(t)f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}\,ds\right|$$
$$= \left|\frac{1}{(n-1)!}\int_a^b T_{w,n}(x,s)\,f^{(n)}(s)\,ds\right| \le \frac{\left\|f^{(n)}\right\|_p}{(n-1)!}\left(\int_a^b |T_{w,n}(x,s)|^q\,ds\right)^{\frac{1}{q}}$$

Let's denote $C_1(s) = \frac{1}{(n-1)!}T_{w,n}(x,s)$. For the proof of the sharpness of the constant $\left(\int_a^b |C_1(s)|^q\,ds\right)^{\frac{1}{q}}$ we will find a function $f$ for which the equality in (3.1) is obtained.

For $1 < p < \infty$ take $f$ to be such that

$$f^{(n)}(s) = sgn\ C_1(s) \cdot |C_1(s)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take

$$f^{(n)}(s) = sgn\ C_1(s).$$

For $p = 1$ we shall prove that

$$\left| \int_a^b C_1(s) f^{(n)}(s) ds \right| \leq \max_{s \in [a,b]} |C_1(s)| \left( \int_a^b \left| f^{(n)}(s) \right| ds \right) \tag{3.2}$$

is the best possible inequality. Suppose that $|C_1(s)|$ attains its maximum at $s_0 \in [a, b]$ First we assume that $C_1(s_0) > 0$. For $\varepsilon$ small enough define $f_\varepsilon(s)$ by

$$f_\varepsilon(s) = \begin{cases} 0, & a \leq s \leq s_0, \\ \dfrac{1}{\varepsilon n!}(s - s_0)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ \dfrac{1}{n!}(s - s_0)^{n-1}, & s_0 + \varepsilon \leq s \leq b. \end{cases}$$

Then, for $\varepsilon$ small enough

$$\left| \int_a^b C_1(s) f^{(n)}(s) ds \right| = \left| \int_{s_0}^{s_0+\varepsilon} C_1(s) \frac{1}{\varepsilon} ds \right| = \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} C_1(s) ds.$$

Now, from inequality (3.2) we have

$$\frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} C_1(s) ds \leq C_1(s_0) \int_{s_0}^{s_0+\varepsilon} \frac{1}{\varepsilon} ds = C_1(s_0).$$

Since,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s_0}^{s_0+\varepsilon} C_1(s) ds = C_1(s_0)$$

the statement follows. In case $C_1(s_0) < 0$, we take

$$f_\varepsilon(s) = \begin{cases} \dfrac{1}{n!}(s - s_0 - \varepsilon)^{n-1}, & a \leq s \leq s_0, \\ -\dfrac{1}{\varepsilon n!}(s - s_0 - \varepsilon)^n, & s_0 \leq s \leq s_0 + \varepsilon, \\ 0, & s_0 + \varepsilon \leq s \leq b, \end{cases}$$

and the rest of proof is the same as above.

**Corollary 1.** *Let $f: I \to \mathbb{R}$ be such that $I \subset \mathbb{R}$ is a open interval, $a, b \in I$, $a < b$ and $(p, q)$ a pair of conjugate exponents, $1 < p \leq \infty$, and $\left|f^{(n)}\right|^p$ on R-integrable function for some $n \geq 2$. Then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt + \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right|$$
$$\leq \frac{1}{n!\,(b-a)} \left( \frac{(x-a)^{qn+1} + (b-x)^{qn+1}}{(nq+1)} \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p$$

*and the constant on the right hand side of the inequality is sharp. For $p = 1$ we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt + \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)} \right|$$
$$\leq \frac{1}{(b-a)\, n!} \max\left\{ (x-a)^n, (b-x)^n \right\} \left\| f^{(n)} \right\|_1$$

*and the constant on the right hand side of the inequality is the best possible.*

**Proof.** We apply the inequality (3.1) with $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ and use (2.2)

$$\int_a^b |T_{w,n}(x,s)|^q\, ds = \int_a^x \left| \frac{-(a-s)^n}{n(b-a)} \right|^q ds + \int_x^b \left| \frac{-(b-s)^n}{n(b-a)} \right|^q ds.$$

First

$$\int_a^x \left| \frac{-(a-s)^n}{n(b-a)} \right|^q ds = \frac{1}{n^q (b-a)^q} \int_a^x (s-a)^{nq}\, ds = \frac{(x-a)^{nq+1}}{n^q (b-a)^q (nq+1)}.$$

Similarly

$$\int_x^b \left| \frac{-(b-s)^n}{n(b-a)} \right|^q ds = \frac{1}{n^q (b-a)^q} \int_x^b (b-s)^{nq}\, ds = \frac{(b-x)^{nq+1}}{n^q (b-a)^q (nq+1)}.$$

So

$$\int_a^b |T_{w,n}(x,s)|^q\, ds = \frac{(x-a)^{qn+1} + (b-x)^{qn+1}}{n^q (b-a)^q (nq+1)}$$

and the first inequality follows from the Theorem 2.

For $p = 1$

$$\sup_{s \in [a,b]} |T_{w,n}(x,s)| = \max\left\{ \sup_{s \in [a,x]} \left| \frac{-(a-s)^n}{n(b-a)} \right|, \sup_{s \in [x,b]} \left| \frac{-(b-s)^n}{n(b-a)} \right| \right\}.$$

By an elementary calculation we get

$$\sup_{s\in[a,x]}\left|\frac{-(a-s)^n}{n(b-a)}\right| = \frac{(x-a)^n}{n(b-a)}$$

and

$$\sup_{s\in[x,b]}\left|\frac{-(b-s)^n}{n(b-a)}\right| = \frac{(b-x)^n}{n(b-a)}.$$

Now, the second inequality follows from the Theorem 2.

**Remark 2.** If we apply (3.1) with $x=\frac{a+b}{2}$ we get the generalized midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)f(t)\,dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{(i+1)!}\int_a^b w(s)\left(s-\frac{a+b}{2}\right)^{i+1}ds\right|$$

$$\leq \frac{1}{(n-1)!}\left(\int_a^b\left|T_{w,n}\left(\frac{a+b}{2},s\right)\right|^q ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.$$

If we additionally assume that $w(t)$ is symmetric on $[a,b]$ i.e. $w(t)=w(b-a-t)$ for every $t\in[a,b]$ this inequality reduces to

$$\left|f\left(\frac{a+b}{2}\right)-\int_a^b w(t)f(t)\,dt+\sum_{i=0}^{\left\lfloor\frac{n}{2}-1\right\rfloor}\frac{f^{(2i)}\left(\frac{a+b}{2}\right)}{(2i)!}2\int_a^{\frac{a+b}{2}} w(s)\left(s-\frac{a+b}{2}\right)^{2i}ds\right|$$

$$\leq \frac{1}{(n-1)!}\left(\int_a^b\left|T_{w,n}\left(\frac{a+b}{2},s\right)\right|^q ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p.$$

For the generalized trapezoid inequality we apply equality (2.1) first with $x=a$, then with $x=b$ then add them up and divide by 2. After applying the Hölder inequality we get

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b w(t)f(t)\,dt\right.$$

$$\left.+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{2(i+1)!}\int_a^b w(s)(s-a)^{i+1}ds+\frac{f^{(i+1)}(b)}{2(i+1)!}\int_a^b w(s)(s-b)^{i+1}ds\right|$$

$$\leq\frac{1}{2(n-1)!}\left(\int_a^b\left|T_{w,n}(a,s)+T_{w,n}(b,s)\right|^q ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

and

$$T_{w,n}(a,s)+T_{w,n}(b,s)=\int_a^s w(u)(u-s)^{n-1}du-\int_s^b w(u)(u-s)^{n-1}du$$

$$=-\int_a^b w(u)|u-s|^{n-1}du$$

Again, if we additionally assume that $w(t)$ is symmetric on $[a,b]$ this inequality reduces to

$$\Big| \frac{f(a)+f(b)}{2} - \int_a^b w(t) f(t)\, dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a) + (-1)^{i+1} f^{(i+1)}(b)}{2(i+1)!} \int_a^b w(s)(s-a)^{i+1} ds \Big| \leq \frac{1}{2(n-1)!} \left( \int_a^b |T_{w,n}(a,s) + T_{w,n}(b,s)|^q ds \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p .$$

## 4. The Estimation of the Difference of the Two Weighted Integral Means

In this section we generalize the results from [1] and [2] We denote

$$t_{w,n}^{[a,b]}(x) = \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)!\,(b-a)}$$

for function $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous function for some $n \geq 2$.

For the two intervals $[a,b]$ and $[c,d]$ we have four possible cases if $[a,b] \cap [c,d] \neq \emptyset$. The first case is $[c,d] \subset [a,b]$ and the second $[a,b] \cap [c,d] = [c,b]$. Other two possible cases we simply get by change $a \leftrightarrow c$, $b \leftrightarrow d$.

**Theorem 3.** *Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be such that $f^{(n-1)}$ is an absolutely continuous function for some $n \geq 2$, $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ some probability density functions, $W(t) = \int_a^t w(x)\,dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$, $U(t) = \int_c^t u(x)\,dx$ for $t \in [c,d]$, $U(t) = 0$ for $t < c$ and $U(t) = 1$ for $t > d$. Then if $[a,b] \cap [c,d] \neq \emptyset$ and $x \in [a,b] \cap [c,d]$, we have*

$$\int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt - t_{w.n}^{[a,b]}(x) + t_{u,n}^{[c,d]}(x) = \int_{\min\{a,c\}}^{\max\{b,d\}} K_n(x,s) f^{(n)}(s)\, ds \tag{4.1}$$

*where in case $[c,d] \subset [a,b]$*

$$K_n(x,s) = \begin{cases} \dfrac{-1}{(n-1)!} \left( \displaystyle\int_a^s w(t)(t-s)^{n-1} dt \right), & s \in [a,c\rangle, \\ \dfrac{-1}{(n-1)!} \left( \displaystyle\int_a^s w(t)(t-s)^{n-1} dt - \int_c^s u(t)(t-s)^{n-1} dt \right), & s \in [c,x], \\ \dfrac{1}{(n-1)!} \left( \displaystyle\int_s^b w(t)(t-s)^{n-1} dt - \int_s^d u(t)(t-s)^{n-1} dt \right), & s \in \langle x,d], \\ \dfrac{1}{(n-1)!} \left( \displaystyle\int_s^b w(t)(t-s)^{n-1} dt \right), & s \in \langle d,b], \end{cases}$$

*and in case* $[a,b]\cap[c,d]=[c,b]$

$$K_n(x,s)=\begin{cases}\dfrac{-1}{(n-1)!}\left(\displaystyle\int_a^s w(t)(t-s)^{n-1}\,dt\right), & s\in[a,c\rangle,\\ \dfrac{-1}{(n-1)!}\left(\displaystyle\int_a^s w(t)(t-s)^{n-1}\,dt-\int_c^s u(t)(t-s)^{n-1}\,dt\right), & s\in[c,x],\\ \dfrac{1}{(n-1)!}\left(\displaystyle\int_s^b w(t)(t-s)^{n-1}\,dt-\int_s^d u(t)(t-s)^{n-1}\,dt\right), & s\in\langle x,b],\\ \dfrac{-1}{(n-1)!}\left(\displaystyle\int_s^d u(t)(t-s)^{n-1}\,dt\right) & s\in\langle b,d],\end{cases}$$

**Proof.** We subtract identities (2.1) for interval $[a,b]$ and $[c,d]$, to get the formula (4.1).

**Theorem 4.** *Assume* $(p,q)$ *is a pair of conjugate exponents, that is* $1\le p,q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\left|f^{(n)}\right|^p:[a,b]\to\mathbb{R}$ *be an R-integrable function for some* $n\ge 2$. *Then we have*

$$\left|\int_a^b w(t)f(t)\,dt-\int_c^d u(t)f(t)\,dt+t_{w,n}^{[a,b]}(x)-t_{u,n}^{[c,d]}(x)\right|$$
$$\le\left(\int_{\min\{a,c\}}^{\max\{b,d\}}|K_n(x,s)|^q\,ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p \tag{4.2}$$

*for every* $x\in[a,b]\cap[c,d]$. *The constant* $\left(\int_{\min\{a,c\}}^{\max\{b,d\}}|K_n(x,s)|^q\,ds\right)^{\frac{1}{q}}$ *is sharp for* $1<p\le\infty$ *and the best possible for* $p=1$.

**Proof.** Use the identity (4.1) and apply the Hölder inequality to obtain

$$\left|\int_a^b w(t)f(t)\,dt-\int_c^d u(t)f(t)\,dt-t_{w,n}^{[a,b]}(x)+t_{u,n}^{[c,d]}(x)\right|$$
$$\le\int_{\min\{a,c\}}^{\max\{b,d\}}|K_n(x,s)|\left|f^{(n)}(s)\right|ds\le\left(\int_{\min\{a,c\}}^{\max\{b,d\}}|K_n(x,s)|^q\,ds\right)^{\frac{1}{q}}\left\|f^{(n)}\right\|_p$$

which proves the inequality. The proof for sharpness and the best possibility are similar as in Theorem 2.

### 4.1. Case $[c,d]\subset[a,b]$

Here we denote

$$t_n^{[a,b]}(x)=\sum_{i=0}^{n-2}f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}.$$

**Corollary 2.** *Let* $\left|f^{(n)}\right| : [a,b] \to \mathbb{R}$ *be an R-integrable function for some* $n \geq 2$, $[c,d] \subset [a,b]$, $x \in [c,d]$, $s_1 = a + \frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $s_2 = b + \frac{d-b}{1-\sqrt[n]{\frac{d-c}{b-a}}}$ *Then if* $s_1 \notin [c,x]$, *and* $s_2 \notin [x,d]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x)\right|$$
$$\leq \frac{1}{(n+1)!\,(b-a)}\left[(x-a)^{n+1} + (b-x)^{n+1} - \frac{b-a}{d-c}(x-c)^{n+1} - \frac{b-a}{d-c}(d-x)^{n+1}\right]\left\|f^{(n)}\right\|_\infty$$

*if* $s_1 \notin [c,x]$, *and* $s_2 \in [x,d]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x)\right|$$
$$\leq \frac{1}{(n+1)!\,(b-a)}\left[(x-a)^{n+1} - (b-x)^{n+1} + 2(b-s_2)^{n+1}\right.$$
$$\left. + \frac{b-a}{d-c}\left((d-x)^{n+1} - (x-c)^{n+1} - 2(d-s_2)^{n+1}\right)\right]\left\|f^{(n)}\right\|_\infty$$

*if* $s_1 \in [c,x]$, *and* $s_2 \notin [x,d]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x)\right|$$
$$\leq \frac{1}{(n+1)!\,(b-a)}\left[-(x-a)^{n+1} + (b-x)^{n+1}\right.$$
$$\left. + 2(s_1-a)^{n+1}\frac{b-a}{d-c}\left((x-c)^{n+1} - (d-x)^{n+1} - 2(s_1-c)^{n+1}\right)\right]\left\|f^{(n)}\right\|_\infty$$

*if* $s_1 \in [c,x]$, *and* $s_2 \in [x,d]$

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x)\right|$$
$$\leq \frac{1}{(n+1)!\,(b-a)}\left[2\left((s_1-a)^{n+1} + (b-s_2)^{n+1}\right) - (x-a)^{n+1} - (b-x)^{n+1}\right.$$
$$\left. + \frac{b-a}{d-c}\left((x-c)^{n+1} + (d-x)^{n+1} - 2\left((s_1-c)^{n+1} + (d-s_2)^{n+1}\right)\right)\right]\left\|f^{(n)}\right\|_\infty$$

*and*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x)\right|$$
$$\leq \frac{1}{n!}\max\left\{\frac{(c-a)^n}{b-a}, \frac{(b-d)^n}{b-a}, \left|\frac{(x-c)^n}{d-c} - \frac{(x-a)^n}{b-a}\right|, \left|\frac{(d-x)^n}{d-c} - \frac{(b-x)^n}{b-a}\right|\right\}\left\|f^{(n)}\right\|_1.$$

**Proof.** We put $w(t) = \frac{1}{b-a}$, $t \in [a,b]$; $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ and $q = 1$ in the Theorem 4. Thus we have $t_n^{[a,b]}(x)$ and $t_n^{[c,d]}(x)$ instead of $t_{w,n}^{[a,b]}(x)$ and $t_{u,n}^{[c,d]}(x)$ and

$$\begin{aligned}
&\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,s)|\, ds \\
&= \frac{1}{(n-1)!}\left(\int_a^c \left|\frac{(a-s)^n}{n(b-a)}\right| ds + \int_c^x \left|\frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)}\right| ds\right. \\
&\left.\int_x^d \left|\frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)}\right| ds + \int_d^b \left|\frac{(b-s)^n}{n(b-a)}\right| ds\right)
\end{aligned}$$

The first and the last integrals are

$$I_1 = \int_a^c \left|\frac{(a-s)^n}{n(b-a)}\right| ds = \frac{1}{n(b-a)}\int_a^c (s-a)^n\, ds = \frac{(c-a)^{n+1}}{n(b-a)(n+1)},$$

$$I_4 = \int_d^b \left|\frac{(b-s)^n}{n(b-a)}\right| ds = \frac{1}{n(b-a)}\int_d^b (b-s)^n\, ds = \frac{(b-d)^{n+1}}{n(b-a)(n+1)},$$

Now, we suppose $n$ is odd. The second integral is

$$\begin{aligned}
I_2 &= \int_c^x \left|\frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)}\right| ds \\
&= \frac{1}{n(b-a)(d-c)}\int_c^x |(d-c)(s-a)^n - (b-a)(s-c)^n|\, ds.
\end{aligned}$$

Let $s_1 = a + \frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $f(s) = (d-c)(s-a)^n - (b-a)(s-c)^n$. We have $f(s_1) = 0$ and $f'(s_1) \neq 0$ so there are two possible cases:

1. If $s_1 > x$, i.e. $s_1 \notin [c,x]$ we have

$$\begin{aligned}
&\int_c^x |(d-c)(s-a)^n - (b-a)(s-c)^n|\, ds \\
&= \int_c^x ((d-c)(s-a)^n - (b-a)(s-c)^n)\, ds \\
&= \frac{1}{n+1}\left((d-c)\left((x-a)^{n+1} - (c-a)^{n+1}\right) - (b-a)(x-c)^{n+1}\right).
\end{aligned}$$

2. If $s_1 < x$, i.e. $s_1 \in [c,x]$ (since $\frac{d-c}{b-a} \leq 1$ so $s_1 > c$)

$$\begin{aligned}
&\int_c^x |(d-c)(s-a)^n - (b-a)(s-c)^n|\, ds \\
&= \int_c^{s_1} ((d-c)(s-a)^n - (b-a)(s-c)^n)\, ds
\end{aligned}$$

$$+\int_{s_1}^{x}\left(-\left(d-c\right)\left(s-a\right)^n+\left(b-a\right)\left(s-c\right)^n\right)ds$$
$$=\frac{1}{n+1}\Big(\left(d-c\right)\Big(2\left(s_1-a\right)^{n+1}-\left(c-a\right)^{n+1}-\left(x-a\right)^{n+1}\Big)$$
$$+\left(b-a\right)\Big(\left(x-c\right)^{n+1}-2\left(s_1-c\right)^{n+1}\Big)\Big).$$

The third integral is

$$I_3=\int_x^d\left|\frac{\left(b-s\right)^n}{n\left(b-a\right)}-\frac{\left(d-s\right)^n}{n\left(d-c\right)}\right|ds$$
$$=\frac{1}{n\left(b-a\right)\left(d-c\right)}\int_x^d\left|\left(d-c\right)\left(b-s\right)^n-\left(b-a\right)\left(d-s\right)^n\right|ds.$$

Let $s_2=b-\frac{b-d}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $g\left(s\right)=\left(d-c\right)\left(b-s\right)^n-\left(b-a\right)\left(d-s\right)^n$. We have $g\left(s_2\right)=0$ and $g'\left(s_2\right)\neq 0$ so again there are two possible cases:

1. If $s_2<x$, i.e. $s_2\notin[x,d]$ (since $s_2<d$) we have

$$\int_x^d\left|\left(d-c\right)\left(b-s\right)^n-\left(b-a\right)\left(d-s\right)^n\right|ds$$
$$=\int_x^d\left(\left(d-c\right)\left(b-s\right)^n-\left(b-a\right)\left(d-s\right)^n\right)ds$$
$$=\frac{1}{n+1}\Big(\left(d-c\right)\Big(\left(b-x\right)^{n+1}-\left(b-d\right)^{n+1}\Big)-\left(b-a\right)\left(d-x\right)^{n+1}\Big).$$

2. If $s_2>x$, i.e. $s_2\in[x,d]$ (since $s_2<d$)

$$\int_x^d\left|\left(d-c\right)\left(b-s\right)^n-\left(b-a\right)\left(d-s\right)^n\right|ds$$
$$=\int_x^{s_2}\left(-\left(d-c\right)\left(b-s\right)^n+\left(b-a\right)\left(d-s\right)^n\right)ds$$
$$+\int_{s_2}^{d}\left(\left(d-c\right)\left(b-s\right)^n-\left(b-a\right)\left(d-s\right)^n\right)ds$$
$$=\frac{1}{n+1}\Big(\left(d-c\right)\Big(-\left(b-x\right)^{n+1}+2\left(b-s_2\right)^{n+1}-\left(b-d\right)^{n+1}\Big)$$
$$+\left(b-a\right)\Big(\left(d-x\right)^{n+1}-2\left(d-s_2\right)^{n+1}\Big)\Big).$$

Now, we suppose $n$ is even. The second integral is

$$I_2=\frac{1}{n\left(b-a\right)\left(d-c\right)}\int_c^x\left|f\left(s\right)\right|ds.$$

Let $s_1=a+\frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$ and $s_3=a+\frac{c-a}{1+\sqrt[n]{\frac{d-c}{b-a}}}$. We have $f\left(s_1\right)=f\left(s_3\right)=0$ and $f'\left(s_1\right)\neq 0$, $f'\left(s_3\right)\neq 0$. By an elementary calculation we also have $s_3<c<s_1$ so there are two possible cases:

1. If $s_3 < c < s_1 < x$

$$\begin{aligned}
&\int_c^x \left|(d-c)(s-a)^n - (b-a)(s-c)^n\right| ds \\
&= \int_c^{s_1} \left((d-c)(s-a)^n - (b-a)(s-c)^n\right) ds \\
&\quad + \int_{s_1}^x \left(-(d-c)(s-a)^n + (b-a)(s-c)^n\right) ds \\
&= \frac{1}{n+1}\Big((d-c)\Big(2(s_1-a)^{n+1} - (c-a)^{n+1} - (x-a)^{n+1}\Big) \\
&\quad + (b-a)\Big((x-c)^{n+1} - 2(s_1-c)^{n+1}\Big)\Big).
\end{aligned}$$

2. If $s_3 < c < x < s_1$

$$\begin{aligned}
&\int_c^x \left|(d-c)(s-a)^n - (b-a)(s-c)^n\right| ds \\
&= \int_c^x \left((d-c)(s-a)^n - (b-a)(s-c)^n\right) ds \\
&= \frac{1}{n+1}\Big((d-c)\Big((x-a)^{n+1} - (c-a)^{n+1}\Big) - (b-a)(x-c)^{n+1}\Big).
\end{aligned}$$

The third integral is

$$I_3 = \frac{1}{n(b-a)(d-c)} \int_x^d |g(s)|\, ds.$$

Let $s_4 = b - \frac{b-d}{1+\sqrt[n]{\frac{d-c}{b-a}}}$ and $s_2 = b - \frac{b-d}{1-\sqrt[n]{\frac{d-c}{b-a}}}$. We have $g(s_2) = g(s_4) = 0$ and $g'(s_2) \neq 0$, $g'(s_4) \neq 0$. By an elementary calculation we also have $s_2 < s_4$ and $d < s_4$ so there are two possible cases:

1. If $s_2 < x < d < s_4$

$$\begin{aligned}
&\int_x^d \left|(d-c)(b-s)^n - (b-a)(d-s)^n\right| ds \\
&= \int_x^d \left((d-c)(b-s)^n - (b-a)(d-s)^n\right) ds \\
&= \frac{1}{n+1}\Big((d-c)\Big((b-x)^{n+1} - (b-d)^{n+1}\Big) - (b-a)(d-x)^{n+1}\Big).
\end{aligned}$$

2. If $x < s_2 < d < s_4$

$$\begin{aligned}
&\int_x^d \left|(d-c)(b-s)^n - (b-a)(d-s)^n\right| ds \\
&= \int_x^{s_2} \left(-(d-c)(b-s)^n + (b-a)(d-s)^n\right) ds
\end{aligned}$$

$$+\int_{s_2}^{d}\left((d-c)(b-s)^n-(b-a)(d-s)^n\right)ds$$
$$=\frac{1}{n+1}\left((d-c)\left(-(b-x)^{n+1}+2(b-s_2)^{n+1}-(b-d)^{n+1}\right)\right.$$
$$\left.+(b-a)\left((d-x)^{n+1}-2(d-s_2)^{n+1}\right)\right).$$

Finally, by summing $I_1$, $I_2$, $I_3$ and $I_4$, the statement for $1<p\leq\infty$ follows.

For $p=1$, by putting $w(t)=\frac{1}{b-a}$, $t\in[a,b]$, and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ in the Theorem 4 again, we have

$$\|K_n(x,s)\|_\infty=\sup_{s\in[a,b]}|K_n(x,s)|$$
$$=\frac{1}{(n-1)!}\max\left\{\max_{s\in[a,c]}\left|\frac{(a-s)^n}{n(b-a)}\right|,\max_{s\in[c,x]}\left|\frac{(a-s)^n}{n(b-a)}-\frac{(c-s)^n}{n(d-c)}\right|\right.$$
$$\left.\max_{s\in[x,d]}\left|\frac{(b-s)^n}{n(b-a)}-\frac{(d-s)^n}{n(d-c)}\right|,\max_{s\in[d,b]}\left|\frac{(b-s)^n}{n(b-a)}\right|\right\}.$$

By an elementary calculation we get

$$\max_{s\in[a,c]}\left|\frac{(a-s)^n}{n(b-a)}\right|=\frac{(c-a)^n}{n(b-a)},\qquad\max_{s\in[d,b]}\left|\frac{(b-s)^n}{n(b-a)}\right|=\frac{(b-d)^n}{n(b-a)}$$

and in both cases $s_1\in[c,x]$ and $s_1\notin[c,x]$

$$\max_{s\in[c,x]}\left|\frac{(a-s)^n}{n(b-a)}-\frac{(c-s)^n}{n(d-c)}\right|=\max\left\{\frac{(c-a)^n}{n(b-a)},\left|\frac{(x-a)^n}{n(b-a)}-\frac{(x-c)^n}{n(d-c)}\right|\right\}$$

and similarly for $s_2\in[x,d]$ and $s_2\notin[x,d]$

$$\max_{s\in[x,d]}\left|\frac{(b-s)^n}{n(b-a)}-\frac{(d-s)^n}{n(d-c)}\right|=\max\left\{\frac{(b-d)^n}{n(b-a)},\left|\frac{(b-x)^n}{n(b-a)}-\frac{(d-x)^n}{n(d-c)}\right|\right\}.$$

Thus, the proof is done.

**Remark 3.** If we put $c=d=x$ as a limit case, the inequalities from Corollary 2 reduce to Ostrowski type inequalities from Corollary 1.

### 4.2. Case $[a,b]\cap[c,d]=[c,b]$

**Corollary 3.** *Let $\left|f^{(n)}\right|:[a,d]\to\mathbb{R}$ be an R-integrable function for some $n\geq 2$, $[a,b]\cap[c,d]\subset[c,b]$, $x\in[c,b]$, $s_1=a+\frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $s_2=b+\frac{d-b}{1-\sqrt[n]{\frac{d-c}{b-a}}}$ Then if $s_1\notin[c,x]$, and $s_2\notin[x,b]$*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(t)\,dt-t_n^{[a,b]}(x)+t_n^{[c,d]}(x)\right|$$
$$\leq\frac{1}{(n+1)!}\left[\frac{(x-a)^{n+1}-(b-x)^{n+1}}{b-a}+\frac{(d-x)^{n+1}-(x-c)^{n+1}}{d-c}\right]\left\|f^{(n)}\right\|_\infty$$

*if* $s_1 \notin [c,x]$, *and* $s_2 \in [x,b]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x) \right|$$

$$\leq \frac{1}{(n+1)!} \left[ \frac{(x-a)^{n+1} + (b-x)^{n+1} - 2(b-s_2)^{n+1}}{(b-a)} + \frac{-(d-x)^{n+1} - (x-c)^{n+1} + 2(d-s_2)^{n+1}}{d-c} \right] \left\| f^{(n)} \right\|_\infty$$

*if* $s_1 \in [c,x]$, *and* $s_2 \notin [x,b]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x) \right|$$

$$\leq \frac{1}{(n+1)!} \left[ \frac{2(s_1-a)^{n+1} - (x-a)^{n+1} - (b-x)^{n+1}}{b-a} + \frac{(x-c)^{n+1} + (d-x)^{n+1} - 2(s_1-c)^{n+1}}{d-c} \right] \left\| f^{(n)} \right\|_\infty$$

*if* $s_1 \in [c,x]$, *and* $s_2 \in [x,b]$

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x) \right|$$

$$\leq \frac{1}{(n+1)!} \left[ \frac{(b-x)^{n+1} - (x-a)^{n+1} + 2(s_1-a)^{n+1} - 2(b-s_2)^{n+1}}{b-a} + \frac{(x-c)^{n+1} - (d-x)^{n+1} + 2(d-s_2)^{n+1} - 2(s_1-c)^{n+1}}{d-c} \right] \left\| f^{(n)} \right\|_\infty$$

*and*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt - t_n^{[a,b]}(x) + t_n^{[c,d]}(x) \right|$$

$$\leq \frac{1}{n!} \max \left\{ \frac{(c-a)^n}{b-a}, \frac{(d-b)^n}{d-c}, \left| \frac{(x-a)^n}{b-a} - \frac{(x-c)^n}{d-c} \right|, \left| \frac{(b-x)^n}{b-a} - \frac{(d-x)^n}{d-c} \right| \right\} \left\| f^{(n)} \right\|_1 .$$

**Proof.** We put $w(t) = \frac{1}{b-a}$, $t \in [a,b]$; $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ and $q = 1$ in the Theorem 4. Thus

$$\int_{\min\{a,c\}}^{\max\{b,d\}} |K_n(x,s)|\,ds = \frac{1}{(n-1)!} \left( \int_a^c \left| \frac{(a-s)^n}{n(b-a)} \right| ds + \int_c^x \left| \frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)} \right| ds \right.$$
$$\left. \int_x^b \left| \frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)} \right| ds + \int_b^d \left| \frac{-(d-s)^n}{n(d-c)} \right| ds \right)$$

The first and the last integrals are

$$I_1 = \int_a^c \left| \frac{(a-s)^n}{n(b-a)} \right| ds = \frac{(c-a)^{n+1}}{n(b-a)(n+1)},$$

$$I_4 = \int_b^d \left| \frac{-(d-s)^n}{n(d-c)} \right| ds = \frac{(d-b)^{n+1}}{n(d-c)(n+1)},$$

Suppose $n$ is odd. The second integral is

$$\begin{aligned} I_2 &= \int_c^x \left| \frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)} \right| ds \\ &= \frac{1}{n(b-a)(d-c)} \int_c^x \left| (d-c)(s-a)^n - (b-a)(s-c)^n \right| ds. \end{aligned}$$

Let $s_1 = a + \frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $f(s) = (d-c)(s-a)^n - (b-a)(s-c)^n$. We have $f(s_1) = 0$ and $f'(s_1) \neq 0$, so there are two possible cases:

1. If $s_1 \notin [c,x]$ we have

$$\begin{aligned} &\int_c^x \left| (d-c)(s-a)^n - (b-a)(s-c)^n \right| ds \\ &= \int_c^x \left( (d-c)(s-a)^n - (b-a)(s-c)^n \right) ds \\ &= \frac{1}{n+1} \left( (d-c) \left( (x-a)^{n+1} - (c-a)^{n+1} \right) - (b-a)(x-c)^{n+1} \right). \end{aligned}$$

2. If $s_1 \in [c,x]$

$$\begin{aligned} &\int_c^x \left| (d-c)(s-a)^n - (b-a)(s-c)^n \right| ds \\ &= \int_c^{s_1} \left( (d-c)(s-a)^n - (b-a)(s-c)^n \right) ds + \int_{s_1}^x \left( -(d-c)(s-a)^n + (b-a)(s-c)^n \right) ds \\ &= \frac{1}{n+1} \Big( (d-c) \left( 2(s_1-a)^{n+1} - (c-a)^{n+1} - (x-a)^{n+1} \right) \\ &\quad + (b-a) \left( (x-c)^{n+1} - 2(s_1-c)^{n+1} \right) \Big). \end{aligned}$$

The third integral is

$$\begin{aligned} I_3 &= \int_x^b \left| \frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)} \right| ds \\ &= \frac{1}{n(b-a)(d-c)} \int_x^b \left| (d-c)(b-s)^n - (b-a)(d-s)^n \right| ds. \end{aligned}$$

Let $s_2 = b + \frac{d-b}{1-\sqrt[n]{\frac{d-c}{b-a}}}$, $g(s) = (d-c)(b-s)^n - (b-a)(d-s)^n$. We have $g(s_2) = 0$ and $g'(s_2) \neq 0$ and again we have two possible cases:

1. If $s_2 \notin [x,b]$ we have

$$\begin{aligned}&\int_x^b \left|(d-c)(b-s)^n-(b-a)(d-s)^n\right| ds\\&=\int_x^b \left(-(d-c)(b-s)^n+(b-a)(d-s)^n\right) ds\\&=\frac{1}{n+1}\left(-(d-c)(b-x)^{n+1}+(b-a)\left((d-x)^{n+1}-(d-b)^{n+1}\right)\right)\end{aligned}$$

2. If $s_2 \in [x,b]$

$$\begin{aligned}&\int_x^b \left|(d-c)(b-s)^n-(b-a)(d-s)^n\right| ds\\&=\int_x^{s_2}\left((d-c)(b-s)^n-(b-a)(d-s)^n\right) ds+\int_{s_2}^b \left(-(d-c)(b-s)^n+(b-a)(d-s)^n\right) ds\\&=\frac{1}{n+1}\left((d-c)\left((b-x)^{n+1}-2(b-s_2)^{n+1}\right)\right.\\&\left.\quad+(b-a)\left(-(d-x)^{n+1}+2(d-s_2)^{n+1}-(d-b)^{n+1}\right)\right)\end{aligned}$$

Now suppose $n$ is even. The second integral is

$$I_2=\frac{1}{n(b-a)(d-c)}\int_c^x |f(s)|\, ds.$$

Let $s_3=a+\frac{c-a}{1+\sqrt[n]{\frac{d-c}{b-a}}}$, $s_1=a+\frac{c-a}{1-\sqrt[n]{\frac{d-c}{b-a}}}$. We have $f(s_1)=f(s_3)=0$ and $f'(s_1)\neq 0$, $f'(s_3)\neq 0$, $s_3<s_1$. By an elementary calculation we also have $a<s_3<c$ so there are three possible cases:

1. If $s_3<s_1<c$ or $s_3<c<x<s_1$ we have

$$\begin{aligned}&\int_c^x \left|(d-c)(s-a)^n-(b-a)(s-c)^n\right| ds\\&=\int_c^x \left((d-c)(s-a)^n-(b-a)(s-c)^n\right) ds\\&=\frac{1}{n+1}\left((d-c)\left((x-a)^{n+1}-(c-a)^{n+1}\right)-(b-a)(x-c)^{n+1}\right).\end{aligned}$$

2. If $s_3<c<s_1<x$

$$\begin{aligned}&\int_c^x \left|(d-c)(s-a)^n-(b-a)(s-c)^n\right| ds\\&=\int_c^{s_1}\left((d-c)(s-a)^n-(b-a)(s-c)^n\right) ds+\int_{s_1}^x \left(-(d-c)(s-a)^n+(b-a)(s-c)^n\right) ds\\&=\frac{1}{n+1}\left((d-c)\left(2(s_1-a)^{n+1}-(c-a)^{n+1}-(x-a)^{n+1}\right)\right.\\&\left.\quad+(b-a)\left((x-c)^{n+1}-2(s_1-c)^{n+1}\right)\right).\end{aligned}$$

The third integral is

$$I_3 = \frac{1}{n(b-a)(d-c)} \int_x^b |g(s)|\, ds.$$

Let $s_4 = b + \frac{d-b}{1+\sqrt[n]{\frac{d-c}{b-a}}}$ $s_2 = b + \frac{d-b}{1-\sqrt[n]{\frac{d-c}{b-a}}}$. We have $g(s_2) = g(s_4) = 0$ and $g'(s_2) \neq 0$, $g'(s_4) \neq 0$. By an elementary calculation we also have $b < s_4$ so there are two possible case:

1. If $x < b < s_4 < s_2$ or $s_2 < x < b < s_4$ we have

$$\begin{aligned} &\int_x^b |(d-c)(b-s)^n - (b-a)(d-s)^n|\, ds \\ &= \int_x^b \left(-(d-c)(b-s)^n + (b-a)(d-s)^n\right) ds \\ &= \frac{1}{n+1}\left(-(d-c)(b-x)^{n+1} + (b-a)\left((d-x)^{n+1} - (d-b)^{n+1}\right)\right). \end{aligned}$$

2. If $x < s_2 < b < s_4$

$$\begin{aligned} &\int_x^b |(d-c)(b-s)^n - (b-a)(d-s)^n|\, ds \\ &= \int_x^{s_2} \left((d-c)(b-s)^n - (b-a)(d-s)^n\right) ds \\ &\quad + \int_{s_2}^b \left(-(d-c)(b-s)^n + (b-a)(d-s)^n\right) ds \\ &= \frac{1}{n+1}\Big((d-c)\left((b-x)^{n+1} - 2(b-s_2)^{n+1}\right) \\ &\quad + (b-a)\left(-(d-x)^{n+1} + 2(d-s_2)^{n+1} - (d-b)^{n+1}\right)\Big). \end{aligned}$$

Finally, by summing $I_1$, $I_2$, $I_3$ and $I_4$, the statement for $1 < p \leq \infty$ follows.

For $p = 1$, by putting $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the Theorem 4 again, we have

$$\begin{aligned} \|K_n(x,s)\|_\infty &= \sup_{s\in[a,b]} |K_n(x,s)| \\ &= \frac{1}{(n-1)!} \max\Bigg\{ \max_{s\in[a,c]} \left|\frac{(a-s)^n}{n(b-a)}\right|, \max_{s\in[c,x]} \left|\frac{(a-s)^n}{n(b-a)} - \frac{(c-s)^n}{n(d-c)}\right| \\ &\quad \max_{s\in[x,b]} \left|\frac{(b-s)^n}{n(b-a)} - \frac{(d-s)^n}{n(d-c)}\right|, \max_{s\in[b,d]} \left|-\frac{(d-s)^n}{n(d-c)}\right| \Bigg\}. \end{aligned}$$

By an elementary calculation we get

$$\max_{s\in[a,c]} \left|\frac{(a-s)^n}{n(b-a)}\right| = \frac{(c-a)^n}{n(b-a)}, \qquad \max_{s\in[b,d]} \left|-\frac{(d-s)^n}{n(d-c)}\right| = \frac{(d-b)^n}{n(d-c)}$$

and in both cases $s_1 \in [c,x]$ and $s_1 \notin [c,x]$

$$\max_{s\in[c,x]}\left|\frac{(a-s)^n}{n(b-a)}-\frac{(c-s)^n}{n(d-c)}\right| = \max\left\{\frac{(c-a)^n}{n(b-a)}, \left|\frac{(x-a)^n}{n(b-a)}-\frac{(x-c)^n}{n(d-c)}\right|\right\}.$$

and similarly for $s_2 \in [x,d]$ and $s_2 \notin [x,d]$

$$\max_{s\in[x,b]}\left|\frac{(b-s)^n}{n(b-a)}-\frac{(d-s)^n}{n(d-c)}\right| = \max\left\{\frac{(d-b)^n}{n(d-c)}, \left|\frac{(b-x)^n}{n(b-a)}-\frac{(d-x)^n}{n(d-c)}\right|\right\}.$$

Thus, the proof is done.

**Remark 4.** If we put $b = c = x$ as a limit case, the inequalities from Corollary 3 reduce to

$$\left|\frac{1}{x-a}\int_a^x f(t)\,dt - \frac{1}{d-x}\int_x^d f(t)\,dt - t_n^{[a,x]}(x) + t_n^{[x,d]}(x)\right|$$
$$\leq \frac{1}{(n+1)!}\left[(x-a)^n + (d-x)^n\right]\left\|f^{(n)}\right\|_\infty$$

and

$$\left|\frac{1}{x-a}\int_a^x f(t)\,dt - \frac{1}{d-x}\int_x^d f(t)\,dt - t_n^{[a,x]}(x) + t_n^{[x,d]}(x)\right|$$
$$\leq \frac{1}{n!}\max\left\{(x-a)^{n-1}, (d-x)^{n-1}\right\}\left\|f^{(n)}\right\|_1.$$

**Remark 5.** If we suppose $b = d$ in both cases $[c,d] \subset [a,b]$ and $[a,b] \cap [c,d] = [c,b]$ the analogues results in Corollary 2 and Corollary 3 coincides.

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Department of Applied Mathematics, Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10 000 Zagreb, Croatia.

E-mail: andrea@zpm.fer.hr

Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.

E-mail: pecaric@hazu.hr