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SOME NEW PROOFS OF THE SUM FORMULA AND RESTRICTED SUM FORMULA

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Abstract. The sum formula is a basic identify of multiple zeta values that expresses a Riemann-zeta value as a homogeneous sum of multiple zeta values of given depth and weight. This formula was already known to Euler in the depth two case. Conjectured in the early 1990s, for higher depth and then proved by Granville and Zagier independently. Restricted sum formula was given in Eie [2]. In this paper, we present some new proofs of those formulas.

1. Introduction

Multiple zeta values are special values of the multi-variable analytic function

$$\zeta(s_1, s_2, \ldots, s_n) = \sum_{1 \le n_1 < n_2 < \cdots < n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

at positive integers $s_i \ge 1$, $s_k \ge 2$, $1 \le i \le k$, the number k and the number $|s| = s_1 + s_2 + \cdots + s_k$ is called the depth and the weight of $\zeta(s_1, s_2, \ldots, s_n)$, respectively. The most well-known sum formula

$$\sum_{\substack{s_i \ge 1,\ \forall i \\ |s| = n}} \zeta(s_1, s_2, \ldots, s_k + 1) = \zeta(n+1)$$

suggesting intriguing connection between Riemann Zeta values and the multiple zeta values. This formula was first obtained by Euler where $k = 2$, known as Euler's sum formula

$$\sum_{i=2}^{n-1} \zeta(n-i, i) = \zeta(n).$$

It's general form was conjectured by M. Schmidt [7] and C. Moen [6] around 1994. A Granville [4] proved the general case in 1997.

For the convenience, we denote $\{1\}^k$ as k repetitions of 1. The sum formula gives $\zeta(\{1\}^{n-1}, 2) = \zeta(n+1)$. In general we have the Drinfeld duality $\zeta(\{1\}^{m-1}, n+1) = \zeta(\{1\}^{n-1}, m+1)$. In this paper,

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we also consider the more general form

$$\sum_{\substack{s_i\geq 2,\ \forall i\\ |s|=m}} \zeta(\{1\}^n, s_1, s_2, \dots, s_k+1)$$

known as the restricted sum formula and give another proof of the restricted sum formula

$$\sum_{\substack{s_i\geq 1,\ \forall i\\ |s|=m}} \zeta(\{1\}^n, s_1, s_2, \dots, s_k+1) = \sum_{\substack{s_i\geq 1,\ \forall i\\ |t|=n+k}} \zeta(t_1, t_2, \dots, t_{p+1}+(m-k)+1)$$

where m, k are position integers with, $m \geq k$ and n is non-negative integer. The restricted sum formula was originally due to M. Eie [2].

2. The double integral representation of a sum

Drinfeld integrals give a multiple zeta values an integral representation. For example

$$\zeta(\{1\}^m, n+1) = \int_{0<t_1<t_2<\cdots<t_{m+n+1}<1} \prod_{j=1}^{m+1} \frac{dt_j}{1-t_j} \prod_{k=m+2}^{m+n+1} \frac{dt_k}{t_k}.$$

In general we have as in [3]

$$\zeta(\alpha_0, \alpha_1, \dots, \alpha_k+1) = \int_{0<t_1<t_2<\cdots<t_{|\alpha|+1}} \Omega_1\Omega_2\cdots\Omega_{|\alpha|+1}$$

where

$$\Omega_k = \begin{cases} \dfrac{dt_k}{1-t_k} & k = 1, \alpha_0+1, \alpha_0+\alpha_1+1, \dots, \alpha_0+\alpha_1+\cdots+\alpha_{k-1}+1, \\ \dfrac{dt_k}{t_k} & \text{otherwise.} \end{cases}$$

For example we have

$$\zeta(2,2,3) = \int_{0<t_1<t_2<\cdots<t_7<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{t_4}\frac{dt_5}{1-t_5}\frac{dt_6}{t_6}\frac{dt_7}{t_7}.$$

Now, fix t_1, $t_{\alpha_0+1}, \dots, t_{\alpha_0+\alpha_1+\cdots+\alpha_{k-1}+1}$ and $t_{|\alpha|+1}$ and integrate with respect to the remainning variables, we obtain that

$$\zeta(\alpha_0, \alpha_1, \dots, \alpha_k+1) = \int_{0<t_1<t_2<\cdots<t_{k+2}<1} \prod_{j=1}^{k+1} \frac{dt_j}{i-t_j} \frac{1}{(\alpha_{j-1}-1)!} \log\left(\frac{t_{j+1}}{t_j}\right)^{\alpha_{j-1}-1} \frac{dt_{k+2}}{t_{k+2}}.$$

Denote $\alpha_j - 1 = \beta_j$, $0 \leq j \leq k$, and consider all the nonnegative integers $\beta_0, \beta_1, \dots, \beta_k$ with $|\beta| = m$, we have

$$\frac{1}{m!}\left(\log\frac{t_{k+2}}{t_1}\right)^m = \frac{1}{m!}\left(\log\frac{t_2}{t_1} + \log\frac{t_3}{t_2} + \cdots + \log\frac{t_{k+2}}{t_{k+1}}\right)^m$$

$$= \sum_{|\beta|=m} \frac{1}{\beta_0!\beta_1!\cdots\beta_k!}\left(\log\frac{t_2}{t_1}\right)^{\beta_0}\left(\log\frac{t_3}{t_2}\right)^{\beta_1}\cdots\left(\log\frac{t_{k+2}}{t_{k+1}}\right)^{\beta_k}.$$

Hence we get

$$\sum_{|\alpha|=m+k+1}\zeta(\alpha_0,\alpha_1,\dots,\alpha_k+1) = \frac{1}{m!}\int_{0<t_1<t_2<\cdots<t_{k+2}<1}\left(\log\frac{t_{k+2}}{t_1}\right)^m\prod_{j=1}^{k+1}\frac{dt_j}{i-t_j}\frac{dt_{k+2}}{t_{k+2}}.$$

Now fixed t_1, t_{k+2}, write it as t_1, t_2, and integrate with respect to the remainning variables we obtain the following theorem.

Theorem 1. *Let m, k be non-negative integers. Then*

$$\sum_{|\alpha|=m+k+1}\zeta(\alpha_0,\alpha_1,\dots,\alpha_k+1) = \frac{1}{m!k!}\int_{0<t_1<t_2<1}\left(\log\frac{1-t_1}{1-t_2}\right)^k\left(\log\frac{t_2}{t_1}\right)^m\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}.$$

Based on the above theorem, we are able to prove the sum formula via changes of variables in the integral.

First proof of the sum formula :

Let

$$x = \log\frac{1}{1-t_1} \qquad y = \log\frac{t_2}{t_1}$$

we have

$$dxdy = \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}, \qquad \log\frac{1-t_1}{1-t_2} = \log\frac{1}{e^x+e^y-e^{x+y}},$$

and

$$\int_{0<t_1<t_2<1}\left(\log\frac{1-t_1}{1-t_2}\right)^k\left(\log\frac{t_2}{t_1}\right)^m\frac{dt_1}{1-t_1}\frac{dt_2}{t_2} = \int_D y^m\left(\log\frac{1}{e^x+e^y-e^{x+y}}\right)^k dxdy$$

where $D = \{(x,y) \mid x>0, y>0, e^x+e^y>e^{x+y}\}$.

On the other hand, also consider the transformation

$$x \to y, \qquad y \to x.$$

Then by the symmetry of D, we get

$$\begin{aligned}\int_D y^m\left(\log\frac{1}{e^x+e^y-e^{x+y}}\right)^k dxdy &= \int_D x^m\left(\log\frac{1}{e^x+e^y-e^{x+y}}\right)^k dxdy\\ &= \int_{0<t_1<t_2<1}\left(\log\frac{1}{1-t_1}\right)^m\left(\log\frac{1-t_1}{1-t_2}\right)^k\frac{dt_1}{1-t_1}\frac{dt_2}{t_2}.\end{aligned}$$

But we already know that

$$\zeta(\{1\}^{m+k},2) = \int_{0<t_1<t_2<\cdots<t_{k+2}<1}\prod_{j=1}^{m+k+1}\frac{dt_j}{1-t_j}\frac{dt_{m+k+2}}{t_{m+k+2}}$$

$$= \frac{1}{m!k!} \int_{0<t_1<t_2<1} \left(\log \frac{1}{1-t_1}\right)^m \left(\log \frac{1-t_1}{1-t_2}\right)^k \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}.$$

So we get

$$\sum_{|\alpha|=m+k+1} \zeta(\alpha_0, \alpha_1, \ldots, \alpha_k + 1) = \zeta(\{1\}^{m+k}, 2) = \zeta(m+k+2)$$

by Drinfeld duality theorem. □

3. Euler sums with two branches

Euler sums with two branches was introduced in Eie [2]. For a pair of integers $p, q \geq 0$, and positive integers $n \geq 2$. We let

$$E_n(p,q) = \sum_{k_0=1}^{\infty} \frac{1}{k_0^n} \sum_k \frac{1}{k_1 k_2 \cdots k_q} \sum_l \frac{1}{l_1 l_2 \cdots l_q},$$

where $(k_0, k_1, \ldots, k_q) \in N^{q+1}$, with $k_0 \geq k_1 \geq \cdots \geq k_q \geq 1$ and $(l_1, \ldots, l_p) \in N^p$, with $k_0 > l_1 > l_2 > \cdots > l_q \geq 1$.

On the other hand we also have

$$E_n(p,q) = \sum_{k \in N^{p+1}} \sum_{u \in N^q} \frac{1}{\sigma_1 \sigma_2 \cdots \sigma_p \sigma_{p+1}^{n-1} \tau_1 \tau_2 \cdots \tau_q (\tau_q + \sigma_{p+1})}$$

where $\sigma_j = k_1 + k_2 + \cdots + k_j$, $\tau_l = u_1 + \cdots + u_l$.

So that we have the following Drinfeld integral representation:

$$E_n(p,q) = \int_D \prod_{j=1}^{p+1} \frac{dt_j}{1-t_j} \prod_{k=p+2}^{n+p-1} \frac{dt_k}{t_k} \prod_{\lambda=1}^{q} \frac{du_\lambda}{1-u_\lambda} \frac{dt_{p+n}}{t_{p+n}},$$

where D is a region of R^{p+q+n} defined by

$$\{0 < t_1 < t_2 < \cdots < t_{p+n} < 1, \quad 0 < u_1 < u_2 < \cdots < u_q < t_{p+n}\}.$$

Now fix t_{p+1}, $_{p+n}$, as t_1, t_2, and integrate with respect to the remainning variables, we get

$$E_n(p,q) = \frac{1}{p!q!(n-2)!} \int_{0<t_1<t_2<1} \left(\log \frac{1}{1-t_1}\right)^p \left(\log \frac{1}{1-t_2}\right)^q \left(\log \frac{t_2}{t_1}\right)^{n-2} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}.$$

Now from the double integral representation for a sum and $E_n(p,q)$, we get

$$\sum_{|\alpha|=m+k+1} \zeta(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = \sum_{p+q=k} (-1)^p E_{m+2}(p,q).$$

We need the following decomposition theorem for $E_{m+2}(p,q)$.

Theorem 2. *For nonnegative integers m, p, q, we have*

$$E_{m+2}(p,q) = \sum_{r=p+1}^{p+q+1} \binom{r-1}{p} \sum_{|\alpha|=p+q+1} \zeta(\alpha_1,\dots,\alpha_r+m+1).$$

We begin with the integral representation.

Proof.

$$E_{m+2}(p,q) = \frac{1}{p!q!m!}\int_{0<t_1<t_2<1} \Big(\log\frac{1}{1-t_1}\Big)^p \Big(\log\frac{1}{1-t_2}\Big)^q \Big(\log\frac{t_2}{t_1}\Big)^m \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}.$$

Express

$$\Big(\log\frac{1}{1-t_2}\Big)^q = \sum_{b=0}^{q}\binom{q}{b}\Big(\log\frac{1}{1-t_1}\Big)^{q-b}\Big(\log\frac{1-t_1}{1-t_2}\Big)^b,$$

the resulting integral represents the sum of multiple zeta value

$$\sum_{b=0}^{q}\binom{p+q-b}{p}\sum_{|\alpha|=b+m+1}\zeta(\{1\}^{p+q-b},\alpha_0,\alpha_1,\dots,\alpha_b+1).$$

Now let $u_1 = 1-t_2$, $u_2 = 1-t_1$, the integral is transformed into

$$\begin{aligned}
&\sum_{b=0}^{q}\frac{1}{p!q!(q-b)!m!}\int_{0<u_1<u_2<1}\Big(\log\frac{1-u_1}{1-u_2}\Big)^m\Big(\log\frac{u_2}{u_1}\Big)^b\Big(\log\frac{1}{u_2}\Big)^{p+q-b}\frac{du_1}{1-u_1}\frac{du_2}{u_2}\\
&=\sum_{b=0}^{q}\binom{p+q-b}{p}\sum_{|\alpha|=b+m+1}\zeta(\alpha_0,\alpha_1,\dots,\alpha_m+p+q-b+1).
\end{aligned}$$

In the final, we apply Ohno's generalization of duality theorem and sum formula with

$$k = (\{1\}^m, p+q-b+2), \quad k' = (\{1\}^{p+q-b}, m+2).$$

We reduce the above sum formula to

$$\sum_{r=p+1}^{p+q+1}\binom{r-1}{p}\sum_{|\alpha|=p+q+1}\zeta(\alpha_1,\alpha_2,\dots,\alpha_r+m+1).$$

New let

$$S_t = \sum_{|\alpha|=r+1}\zeta(\alpha_1,\dots,\alpha_t+m+1), \quad t = 1,2,\dots,r+1.$$

we get the following from the decomposition theorem

$$\begin{aligned}
E_{m+2}(0,r) &= S_{r+1}+S_r+\cdots+S_3+s_2+s_1\\
E_{m+2}(1,r-1) &= rS_{r+1}+(r-1)S_r+\cdots+2S_3+s_2\\
E_{m+2}(2,r-2) &= \binom{r}{2}S_{r+1}+\binom{r-1}{2}S_r+\cdots+S_3
\end{aligned}$$

$$\vdots$$

$$E_{m+2}(r,0) = S_{r+1}.$$

So we get $\sum\limits_{p+q=r}(-1)^p E_{m+2}(p,q) = S_1 = \zeta(m+r+2)$. This gives a second proof of the sum formula. □

4. The restricted sum formula

The restricted sum can be expressed as an double integral:

$$\sum_{|\alpha|=q+r+1} \zeta(\{1\}^p, \alpha_0, \alpha_1, \ldots, \alpha_q + 1)$$
$$= \frac{1}{p!q!r!}\int_{0<t_1<t_2<1}\Big(\log\frac{1}{1-t_1}\Big)^p\Big(\log\frac{1-t_1}{1-t_2}\Big)^q\Big(\log\frac{t_2}{t_1}\Big)^r \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}.$$

Make the change of variables:

$$x = \log\frac{1}{1-t_1} \qquad y = \log\frac{t_2}{t_1}$$

the above integral is transformed into

$$\frac{1}{p!q!r!}\int_{D_2} x^p y^r \Big(\frac{1}{e^x + e^y - e^{x+y}}\Big)^q dx dy.$$

Change coordinate again $x = \log\frac{1-u_1}{1-u_2}$, $y = \log\frac{1}{u_2}$.

We get the following integral representation for the restricted sum:

$$\frac{1}{p!q!r!}\int_{0<u_1<u_2<1}\Big(\log\frac{1-u_1}{1-u_2}\Big)^p\Big(\log\frac{u_2}{u_1}\Big)^q\Big(\log\frac{1}{u_2}\Big)^r \frac{du_1}{1-u_1}\frac{du_2}{u_2}$$

which is

$$\sum_{|c|=p+q+1} \zeta(c_0, c_1, \ldots, c_p + r + 1).$$

Therefore, the restricted sum formula is proved. In the following, we given another proof of the same formula. From the double integral formula for both

$$\sum_{|\alpha|=q+r+1} \zeta(\{1\}^p, \alpha_0, \alpha_1, \ldots, \alpha_q + 1)$$

and $E_m(p,q)$ we get

$$\sum_{|\alpha|=q+r+1} \zeta(\{1\}^p, \alpha_0, \alpha_1, \ldots, \alpha_q + 1) = \sum_{b=0}^{q}\binom{p+b}{b}(-1)^b E_{r+2}(p+b, q-b).$$

Substitute the decomposition of $E_{r+2}(p+b,q-b)$ as

$$\sum_{g=p+b+1}^{p+q+1}\binom{g-1}{p+b}S_g,\ \text{with } S_g=\sum_{|\alpha|=p+q+1}\zeta(\alpha_1,\dots,\alpha_g+r+1)$$

into the identity

$$\sum_{|\alpha|=q+r+1}\zeta(\{1\}^p,\alpha_0,\alpha_1,\dots,\alpha_q+1)=\sum_{b=0}^{q}\binom{p+b}{b}(-1)^b E_{r+2}(p+b,q-b).$$

We get

$$\sum_{|\alpha|=q+r+1}\zeta(\{1\}^p,\alpha_0,\dots,\alpha_{p+1})=\sum_{b=0}^{q}\binom{p+b}{p}(-1)^b\sum_{g=p+b+1}^{p+q+1}\binom{g-1}{p+b}S_g.$$

Exchange the order of summation, we get

$$\sum_{g=p+1}^{p+q+1}\binom{g-1}{p}S_g\sum_{b=0}^{g-p-1}\binom{g-p-1}{b}(-1)^b.$$

Again, the inner sum is zero unless $g-p-1=0$. Therefore the total sum is S_{p+1} or $\sum_{|c|=p+q+1}\zeta(c_0,c_1,\dots,c_p+r+1)$.

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