GENERALIZATION OF AN INEQUALITY OF ALZER FOR NEGATIVE POWERS

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Abstract. Let \( \{a_n\}_{n=1}^{\infty} \) be a positive, strictly increasing, and logarithmically concave sequence satisfying
\[
\left( \frac{a_{n+1}}{a_n} \right)^n < \left( \frac{a_{n+2}}{a_{n+1}} \right)^{n+1}.
\]
Then we have
\[
\frac{a_n}{a_{n+m}} < \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r / \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r},
\]
where \( n, m \) are natural numbers and \( r \) is a positive real number. The lower bound is the best possible. This generalizes an inequality of Alzer for negative powers.

1. Introduction

When studying a problem on upper bound for permanents of \((0,1)\)-matrices, in 1964 H. Minc and L. Sathre [5] discovered several noteworthy inequalities involving \( (n!)^{1/n} \). One of them is the following: If \( n \) is a positive integer, then
\[
\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{n+1} \leq 1.
\]
(1)

By investigating a problem on Lorentz sequence spaces, in 1988 J. S. Martins [4] published another lower bound for \( \sqrt[n]{n!}/n!^{1/(n+1)} \): Let \( r \) be a positive real number and let \( n \) be a natural number, then
\[
\left( \frac{1}{n} \sum_{i=1}^{n} i^r / \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r} \leq \frac{\sqrt[n]{n!}}{n!^{1/(n+1)}}.
\]
(2)
In 1993 H. Alzer [1] compared the lower bounds of (1) and (2), and established the following result: Let $n$ be a positive integer, then for any positive real number $r$,

$$
\frac{n}{n + 1} \leq \left( \frac{1}{n} \sum_{i=1}^{n} i^r \right) / \left( \frac{1}{n + 1} \sum_{i=1}^{n+1} i^r \right)^{1/r}.
$$

The proof given by Alzer is remarkable, but it is quite long and complicated. Several easy proofs of (3) have been published by different authors, see [2, 7, 8], and these proofs show that in fact (3) holds with strictly inequality. By mathematical induction and Cauchy’s mean-value theorem, F. Qi [6] generalized the inequality (3) and showed that: Let $n$ and $m$ be natural numbers, $k$ a nonnegative integer, then

$$
\frac{n + k}{n + m + k} < \left( \frac{1}{n} \sum_{i=k+1}^{n+k} i^r \right) / \left( \frac{1}{n + m} \sum_{i=k+1}^{n+m+k} i^r \right)^{1/r},
$$

where $r$ is any given positive real number. The lower bound is the best possible. In fact, (4) is essentially equivalent to

$$
\frac{n}{n + m} < \left( \frac{1}{n} \sum_{i=1}^{n} a^r \right) / \left( \frac{1}{n + m} \sum_{i=1}^{n+m} a^r \right)^{1/r}.
$$

In this paper, the inequalities (3) and (5) are further generalized as follows.

**Theorem.** Let $\{a_n\}_{n=1}^{\infty}$ be a positive and strictly increasing sequence satisfying

$$
\frac{a_n}{a_{n+1}} \leq \frac{a_{n+1}}{a_{n+2}}, \quad n \in \mathbb{N} := \{1, 2, \ldots\}
$$

and

$$
\left( \frac{a_{n+1}}{a_n} \right)^n \leq \left( \frac{a_{n+2}}{a_{n+1}} \right)^{n+1}, \quad n \in \mathbb{N}.
$$

Then we have

$$
\frac{a_n}{a_{n+m}} < \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r \right) / \left( \frac{1}{n + m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r},
$$

where $n, m \in \mathbb{N}$ and $r$ is a positive real number. The lower bound is the best possible.

Notice that if a positive sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the inequality (6), then we call it a logarithmically concave sequence.

**Proof.** The inequality (8) can be written as

$$
\frac{1}{(n + m)a_{n+m}} \sum_{i=1}^{n+m} a_i^r \leq \frac{1}{na_n^r} \sum_{i=1}^{n} a_i^r,
$$
which is equivalent to

\[
\frac{1}{(n+1)a_{n+1}^r} \sum_{i=1}^{n+1} a_i^r < \frac{1}{na_n^r} \sum_{i=1}^{n} a_i^r. \tag{9}
\]

Since

\[
\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^{n} a_i^r + a_{n+1}^r,
\]

(9) reduces to

\[
\sum_{i=1}^{n} a_i^r > \frac{na_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r}. \tag{10}
\]

It is easy to see that the inequality (10) holds for \( n = 1 \). Suppose that the inequality (10) holds for some \( n = k (k \geq 1) \), that is

\[
\sum_{i=1}^{k} a_i^r > \frac{ka_k^r a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r}. \tag{11}
\]

Adding \( a_{k+1}^r \) to the both sides of (11), we have

\[
\sum_{i=1}^{k+1} a_i^r > \frac{(k+1)a_{k+1}^r a_{k+2}^r}{(k+1)a_{k+2}^r - (k+1)a_{k+1}^r}. \tag{12}
\]

By mathematical induction, it remains to show that

\[
\sum_{i=1}^{k+1} a_i^r > \frac{(k+1)a_{k+1}^r a_{k+2}^r}{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}. \tag{13}
\]

From (12) and (13) it is sufficient to show that

\[
\frac{(k+1)a_{k+1}^r a_{k+2}^r}{(k+1)a_{k+1}^r - ka_k^r} > \frac{(k+1)a_k^r a_{k+2}^r}{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r},
\]

which can be rearranged as

\[
(k+1) \left( \frac{a_{k+1}}{a_{k+2}} \right)^r - k \left( \frac{a_k}{a_{k+1}} \right)^r < 1. \tag{14}
\]

We define for \( r > 0 \)

\[
f(r) = (k+1) \left( \frac{a_{k+1}}{a_{k+2}} \right)^r - k \left( \frac{a_k}{a_{k+1}} \right)^r.
\]

Differentiation yields

\[
f'(r) = (k+1) \left( \frac{a_{k+1}}{a_{k+2}} \right)^r \ln \left( \frac{a_{k+1}}{a_{k+2}} \right) - k \left( \frac{a_k}{a_{k+1}} \right)^r \ln \left( \frac{a_k}{a_{k+1}} \right) \]

\[
= - \left( \frac{a_{k+1}}{a_{k+2}} \right)^r \ln \left( \frac{a_{k+2}}{a_{k+1}} \right)^{k+1} + \left( \frac{a_k}{a_{k+1}} \right)^r \ln \left( \frac{a_{k+1}}{a_k} \right)^k.
\]
Since
\[ 0 < \frac{a_k}{a_{k+1}} \leq \frac{a_{k+1}}{a_{k+2}} \quad k \in \mathbb{N}, \]
\[ 1 < \left( \frac{a_{k+1}}{a_k} \right)^k < \left( \frac{a_{k+2}}{a_{k+1}} \right)^{k+1}, \quad k \in \mathbb{N}. \]

It is easy to see that
\[ \left( \frac{a_k}{a_{k+1}} \right)^r \ln \left( \frac{a_{k+1}}{a_k} \right)^k < \left( \frac{a_{k+1}}{a_{k+2}} \right)^r \ln \left( \frac{a_{k+2}}{a_{k+1}} \right)^{k+1} \]
which implies that \( f'(r) < 0 \) and \( f(r) < f(0) = 1 \), and then (14) holds.

By L’ Hospital rule, easy calculation produces
\[ \lim_{r \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} a_i^r \right) \left( \frac{1}{n + m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r} = \frac{a_n}{a_{n+m}}, \]
thus, the lower bound given in (8) is the best possible. The proof is complete.

The authors [3] showed that (3) holds strictly for all natural numbers \( n \) and all real numbers \( r \). Now we pose the following open problem.

**Open Problem.** What conditions does the sequence \( \{a_k\}_{k=1}^{\infty} \) satisfy such that (8) holds for all natural numbers \( n, m \) and all real numbers \( r \)?

**References**


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