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GENERALIZATION OF AN INEQUALITY OF ALZER FOR NEGATIVE POWERS

CHAO-PING CHEN AND FENG QI

Abstract. Let $\{a_n\}_{n=1}^{\infty}$ be a positive, strictly increasing, and logarithmically concave sequence satisfying $(a_{n+1}/a_n)^n < (a_{n+2}/a_{n+1})^{n+1}$. Then we have

$$\frac{a_n}{a_{n+m}} < \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r},$$

where n, m are natural numbers and r is a positive real number. The lower bound is the best possible. This generalizes an inequality of Alzer for negative powers.

1. Introduction

When studying a problem on upper bound for permanents of $(0,1)$–matrices, in 1964 H. Minc and L. Sathre [5] discovered several noteworthy inequalities involving $(n!)^{1/n}$. One of them is the following: If n is a positive integer, then

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1. \tag{1}$$

By investigating a problem on Lorentz sequence spaces, in 1988 J. S. Martins [4] published another lower bound for $\sqrt[n]{n!}/\sqrt[n+1]{(n+1)!}$: Let r be a positive real number and let n be a natural number, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} \le \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{2}$$

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In 1993 H. Alzer [1] compared the lower bounds of (1) and (2), and established the following result: Let n be a positive integer, then for any positive real numbe r,

$$\frac{n}{n+1} \leq \left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{1/r}. \tag{3}$$

The proof given by Alzer is remarkable, but it is quite long and complicated. Several easy proofs of (3) have been published by different authors, see [2, 7, 8], and these proofs show that in fact (3) holds with strictly inequality. By mathematical induction and Cauchy's mean-value theorem, F. Qi [6] generalized the inequality (3) and showed that: Let n and m be natural numbers, k a nonnegative integer, then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n} \sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r \right)^{1/r}, \tag{4}$$

where r is any given positive real number. The lower bound is the best possible. In fact, (4) is essentially equivalent to

$$\frac{n}{n+m} < \left(\frac{1}{n} \sum_{i=1}^{n} i^r \Big/ \frac{1}{n+m} \sum_{i=1}^{n+m} i^r \right)^{1/r}. \tag{5}$$

In this paper, the inequalities (3) and (5) are further generalized as follows.

Theorem. *Let $\{a_n\}_{n=1}^{\infty}$ be a positive and strictly increasing sequence satisfying*

$$\frac{a_n}{a_{n+1}} \leq \frac{a_{n+1}}{a_{n+2}}, \quad n \in \mathbb{N} := \{1, 2, \dots\} \tag{6}$$

and

$$\left(\frac{a_{n+1}}{a_n} \right)^n < \left(\frac{a_{n+2}}{a_{n+1}} \right)^{n+1}, \quad n \in \mathbb{N}. \tag{7}$$

Then we have

$$\frac{a_n}{a_{n+m}} < \left(\frac{1}{n} \sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r}, \tag{8}$$

where $n, m \in \mathbb{N}$ and r is a positive real number. The lower bound is the best possible.

Notice that if a positive sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the inequality (6), then we call it a logarithmically concave sequence.

Proof. The inequality (8) can be written as

$$\frac{1}{(n+m)a_{n+m}^r} \sum_{i=1}^{n+m} a_i^r < \frac{1}{n a_n^r} \sum_{i=1}^{n} a_i^r,$$

which is equivalent to

$$\frac{1}{(n+1)a_{n+1}^r}\sum_{i=1}^{n+1} a_i^r < \frac{1}{na_n^r}\sum_{i=1}^{n} a_i^r. \tag{9}$$

Since

$$\sum_{i=1}^{n+1} a_i^r = \sum_{i=1}^{n} a_i^r + a_{n+1}^r,$$

(9) reduces to

$$\sum_{i=1}^{n} a_i^r > \frac{na_n^r a_{n+1}^r}{(n+1)a_{n+1}^r - na_n^r}. \tag{10}$$

It is easy to see that the inequality (10) holds for $n = 1$. Suppose that the inequality (10) holds for some $n = k (k \geq 1)$, that is

$$\sum_{i=1}^{k} a_i^r > \frac{ka_k^r a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r}. \tag{11}$$

Adding a_{k+1}^r to the both sides of (11), we have

$$\sum_{i=1}^{k+1} a_i^r > \frac{(k+1)a_{k+1}^{2r}}{(k+1)a_{k+1}^r - ka_k^r}. \tag{12}$$

By mathematical induction, it remains to show that

$$\sum_{i=1}^{k+1} a_i^r > \frac{(k+1)a_{k+1}^r a_{k+2}^r}{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}. \tag{13}$$

From (12) and (13) it is sufficient to show that

$$\frac{(k+1)a_{k+1}^{2r}}{(k+1)a_{k+1}^r - ka_k^r} > \frac{(k+1)a_{k+1}^r a_{k+2}^r}{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r},$$

which can be rearranged as

$$(k+1)\left(\frac{a_{k+1}}{a_{k+2}}\right)^r - k\left(\frac{a_k}{a_{k+1}}\right)^r < 1. \tag{14}$$

We difine for $r > 0$

$$f(r) = (k+1)\left(\frac{a_{k+1}}{a_{k+2}}\right)^r - k\left(\frac{a_k}{a_{k+1}}\right)^r.$$

Differentiation yields

$$\begin{aligned} f'(r) &= (k+1)\left(\frac{a_{k+1}}{a_{k+2}}\right)^r \ln\left(\frac{a_{k+1}}{a_{k+2}}\right) - k\left(\frac{a_k}{a_{k+1}}\right)^r \ln\left(\frac{a_k}{a_{k+1}}\right) \\ &= -\left(\frac{a_{k+1}}{a_{k+2}}\right)^r \ln\left(\frac{a_{k+2}}{a_{k+1}}\right)^{k+1} + \left(\frac{a_k}{a_{k+1}}\right)^r \ln\left(\frac{a_{k+1}}{a_k}\right)^k. \end{aligned}$$

Since

$$0 < \frac{a_k}{a_{k+1}} \leqslant \frac{a_{k+1}}{a_{k+2}}, \quad k \in \mathbb{N},$$

$$1 < \left(\frac{a_{k+1}}{a_k}\right)^k < \left(\frac{a_{k+2}}{a_{k+1}}\right)^{k+1}, \quad k \in \mathbb{N}.$$

It is easy to see that

$$\left(\frac{a_k}{a_{k+1}}\right)^r \ln\left(\frac{a_{k+1}}{a_k}\right)^k < \left(\frac{a_{k+1}}{a_{k+2}}\right)^r \ln\left(\frac{a_{k+2}}{a_{k+1}}\right)^{k+1}$$

which implies that $f'(r) < 0$ and $f(r) < f(0) = 1$, and then (14) holds.

By L' Hospital rule, easy caculation produces

$$\lim_{r\to+\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} = \frac{a_n}{a_{n+m}},$$

thus, the lower bound given in (8) is the best possible. The proof is complete.

The authors [3] showed that (3) holds strictly for all natural numbers n and all real numbers r. Now we pose the following open problem.

Open Problem. What conditions does the sequence $\{a_k\}_{k=1}^{\infty}$ satisfy such that (8) holds for all natural numbers n, m and all real numbers r?

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Department of Applied Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan 454010, CHINA.

E-mail: chenchaoping@sohu.com

Department of Applied Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan 454010, CHINA.

E-mail: qifeng@jzit.edu.cn