SOME INEQUALITIES OF SLANT SUBMANIFOLDS IN 
GENERALIZED COMPLEX SPACE FORMS

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Abstract. In this paper, we obtain an inequality about Ricci curvature and squared mean 
curvature of slant submanifolds in generalized complex space forms. We also obtain an inequality 
about the squared mean curvature and the normalized scalar curvature of slant submanifolds in 
generalized complex space forms.

1. Introduction

To find simple relationships between the main intrinsic invariants and the main extrinsic 
invariants of a submanifold is one of the basic interests of study in the submanifold 
theory. Scalar curvature and Ricci curvature are among the main intrinsic invariants, while 
the squared mean curvature is the main extrinsic invariants. In [5], an inequality about 
the Ricci curvature and the squared mean curvature of a slant submanifold in complex 
space forms was obtained by K. Matsumoto, I. Mihai and Y. Tazawa. Also a sharp 
relationship between the normalized scalar curvature and the squared mean curvature of 
isometric in real-space-forms has been proofed by B. Y. Chen in [2].

On the other hand, the definition of the generalized complex space forms was intro-
duced by F. Tricerri and L. Vanhecke in [4]. In the present paper, we study the slant 
submanifolds in generalized complex space forms, and we obtain an inequality about 
Ricci curvature and squared mean curvature of slant submanifolds in generalized com-
plex space forms. Also we get an inequality about the squared mean curvature and the 
normalized scalar curvature of slant submanifolds in generalized complex space forms. 
These results generalize the corresponding results in [5] and [2].

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with an almost Hermitian structure $(J, \langle.,.\rangle)$. 
An almost Hermitian manifold becomes a nearly Kähler manifold [1] if $(\tilde{\nabla}_X J)X = 0$, 
and become a Kähler manifold if $\tilde{\nabla}J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is the levi-Civita 

Received and revised April 28, 2004.
2000 Mathematics Subject Classification. 53C42, 53C40.
Key words and phrases. Ricci curvature, Ricci tensor, squared mean curvature, generalized complex space forms, slant submanifolds.
connection of the Riemannian metric $\langle \cdot, \cdot \rangle$. An almost Hermitian manifold with $J$-invariant Riemannian curvature tensor $\tilde{R}$, that is,
\[
\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W), \quad X, Y, Z, W \in T\tilde{M}
\]
is called an RK-manifolds [4]. All nearly Kähler manifolds belong to the class of RK-manifolds.

The notion of constant type was first introduced by A. Gray for a nearly Kähler manifold $\tilde{M}$ is said to have (pointwise) constant type if for each $p \in \tilde{M}$ and for $X, Y, Z \in T_p\tilde{M}$ such that
\[
\langle X, Y \rangle = \langle X, Z \rangle = \langle X, JY \rangle = \langle X, JZ \rangle = 0, \quad \langle Y, Y \rangle = 1 = \langle Z, Z \rangle
\]
we have
\[
\tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \tilde{R}(X, Z, X, Z) - \tilde{R}(X, Z, JX, JZ).
\]

An RK-manifold $\tilde{M}$ has pointwise constant type if and only if there is a differentiable function $\alpha$ on $\tilde{M}$ satisfying [6]
\[
4\tilde{R}(X, Y)Z = (c + 3\alpha)\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\} + (c - \alpha)\{\langle X, JZ \rangle JY - \langle Y, JZ \rangle JX\}
\]
for all $X, Y, Z \in T\tilde{M}$. Furthermore, $\tilde{M}$ has global constant type if $\alpha$ is constant. The function $\alpha$ is called the constant type of $\tilde{M}$.

An RK-manifold of constant holomorphic sectional curvature $c$ and constant type $\alpha$ is denoted by $\tilde{M}(c, \alpha)$. For $\tilde{M}(c, \alpha)$ it is known that [6]
\[
4\tilde{R}(X, Y)Z = f_1\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\} + f_2\{\langle X, JZ \rangle JY - \langle Y, JZ \rangle JX + 2\langle X, JY \rangle JZ\}
\]
for all $X, Y, Z \in T\tilde{M}$. If $c = \alpha$ then $\tilde{M}(c, \alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c, \alpha)$ (a Kähler manifold of constant holomorphic sectional curvature $c$) belongs to the class of almost Hermitian manifolds $\tilde{M}(c, \alpha)$ (with the constant type zero).

An almost Hermitian manifold $\tilde{M}$ is called a generalized complex space form $\tilde{M}(f_1, f_2)$ [4] if its Riemannian curvature tensor $\tilde{R}$ satisfies
\[
\tilde{R}(X, Y)Z = f_1\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\} + f_2\{\langle X, JZ \rangle JY - \langle Y, JZ \rangle JX + 2\langle X, JY \rangle JZ\}
\]
for all $X, Y, Z \in T\tilde{M}$, where $f_1$ and $f_2$ are smooth functions on $\tilde{M}$.

Let $M$ be $n$-dimensional submanifold of an $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Let $p \in M$ and $\{e_1, \ldots, e_{2m}\}$ an orthonormal basis at $p$, such that $\{e_1, \ldots, e_n\}$ are tangent to $M$ and $e_{n+1}, \ldots, e_{2m}$ are normal to $M$. We denote the sectional curvature of the plane section $\pi \in T_p\tilde{M}$ by $K(\pi)$. For an orthonormal basis
$e_1, \ldots, e_n$ of the tangent space $T_pM$, the scalar curvature $\pi$ and the normalized scalar curvature $\rho$ are defined respectively by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \quad \rho = \frac{2\tau}{n(n-1)}.$$  

Then the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))$$  

for any vectors $X, Y, Z, W$ tangent to $M$, where $R$ is the Riemannian curvature tensor of $M$.

We denote by $H$ the mean curvature vector at $p \in M$, i.e.

$$H(P) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, \ldots, e_{2m}\}$ is an orthonormal basis of the tangent space $T_p\tilde{M}(f_1, f_2)$, such that $\{e_1, \ldots, e_n\}$ are tangent to $M$.

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j = 1, \ldots, n; \quad r = n+1, \ldots, 2m,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any $p \in M$ and for any $X \in T_pM$, we put $JX = PX + FX$, where $PX \in T_pM$, $FX \in T^\perp_pM$.

We put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is said to be a slant submanifold if for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between $JX$ and the tangent space $T_pM$ is constant ($= \theta$). $M$ is called a totally real submanifold if the almost complex structure $J$ of $\tilde{M}$ carries each tangent space of $M^n$ into its corresponding normal space (i.e. $\theta = \pi/2$).

If $M$ is a $\theta$-slant submanifold in a generalized complex space form $\tilde{M}(f_1, f_2)$, then

$$\|P\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g^2(Pe_i, e_j) = \sum_{i=1}^{n} \|Pe_i\|^2 = n \cos^2 \theta.$$  

We recall that for a submanifold $M$ in a Riemannian manifold, the relative null space of $M$ at a point $p$ is defined by

$$N_p = \{ X \in T_pM \mid h(X, Y) = 0, \text{ for all } Y \in T_pM \}.$$
3. Ricci Tensor and Squared Mean Curvature

Let $M$ be a submanifold in a generalized complex space form, then the Gauss equation becomes [7]

$$R(X, Y, Z, W) = f_1 \langle \langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle \rangle + f_2 \langle \langle X, PZ \rangle \langle PY, W \rangle - \langle Y, PZ \rangle \langle PX, W \rangle + 2 \langle X, PY \rangle \langle PZ, W \rangle \rangle + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle$$

for all $X, Y, Z, W \in TM$. Thus, we are able to state the following lemma.

**Lemma 2.1.** For an $n$-dimensional submanifold in a generalized complex space form $\tilde{M}(f_1, f_2)$, the scalar curvature and the squared mean curvature satisfy

$$2\tau = n(n-1)f_1 + 3f_2 ||P||^2 + n^2 ||H||^2 - ||h||^2.$$ (1)

First we will prove the following basic inequality. It generalizes Theory 2.1 in [5].

**Theorem 2.2.** Let $M$ be an $n$-dimensional $\theta$-slant submanifold in an $m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then:

i) For each unit vector $X \in T_pM$, we have

$$\text{Ric}(X) \leq \frac{n^2}{4} ||H||^2 + (n-1)f_1 + 3f_2 \cos^2 \theta.$$ (2)

ii) If $H(p) = 0$, then a unit tangent vector $X$ at $p$ satisfies the equality case of (2.1) if and only if $X \in N_p$.

iii) The equality case of (2) holds identically for all unit tangent vectors at $p$ if and only if either $p$ is a totally geodesic point or $n = 2$ and $p$ is a totally umbilical point.

**Proof.** Let $X \in T_pM$ be a unit tangent vector at $p$. We choose an orthogonal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}$ such that $e_1, \ldots, e_n$ are tangent to $M$ at $p$, with $e_1 = X$. Then from (1), we have

$$n^2 ||h||^2 = 2\tau + ||h||^2 - n(n-1)f_1 - 3nf_2 \cos^2 \theta.$$ (3)

Form (3) we get

$$n^2 ||H||^2 = 2\tau + \sum_{r=n+1}^{2m} \left[ (h^r_{11})^2 + (h^r_{22})^2 + \cdots + (h^r_{nn})^2 + 2 \sum_{i<j}(h^r_{ij})^2 \right]$$

$$+ 2 \sum_{r=n+1}^{2m} \sum_{2 \leq i < j \leq n} h^r_{ii}h^r_{jj} - [n(n-1)f_1 + 3nf_2 \cos^2 \theta]$$

$$= 2\tau + \frac{1}{2} \sum_{r=n+1}^{2m} \left[ ((h^r_{11} + \cdots + h^r_{nn})^2 + (h^r_{11} - h^r_{22} - \cdots - h^r_{nn})^2) \right]$$

$$+ 2 \sum_{r=n+1}^{2m} \sum_{i<j}(h^r_{ij})^2 - 2 \sum_{r=n+1}^{2m} \sum_{2 \leq i < j \leq n} h^r_{ii}h^r_{jj} - [n(n-1)f_1 + 3nf_2 \cos^2 \theta].$$ (4)
From the equation of Gauss, we have

\[ K_{ij} = 2m \sum_{r=n+1}^{2m} \left[ h^r_{ir} h^r_{jj} - (h^r_{ij})^2 \right] + f_1 + 3f_2g^2(Pe_i, e_j). \]  

(5)

and consequently

\[ \sum_{2 \leq i < j \leq n} K_{ij} = 2m \sum_{r=n+1}^{n+1} \sum_{2 \leq i < j \leq n} \left[ h^r_{ir} h^r_{jj} - (h^r_{ij})^2 \right] + \frac{(n-1)(n-2)}{2} f_1 + \frac{3}{2} n f_2 \cos^2 \theta - 3f_2 \cos^2 \theta. \]  

(6)

Substituting (6) in (4), we have

\[ n^2 \| H \|^2 \geq 2r + \frac{1}{2} n^2 \| H \|^2 + 2 \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (h^r_{ij})^2 - 2 \sum_{2 \leq i < j \leq n} K_{ij} \]

\[ + [(n-1)(n-2) - n(n-1)] f_1 - 6f_2 \cos^2 \theta. \]  

(7)

From inequality (7) we could get (2).

ii) Assume \( H(p) = 0 \), equality holds in (2) if and only if

\[ \begin{cases} h^r_{ij} = 0, & i \neq j, r \in \{ n+1, \ldots, 2m \}, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{nn}, & r \in \{ n+1, \ldots, 2m \}. \end{cases} \]

Then \( h^r_{ij} = 0 \) \( \forall j \in \{ 1, \ldots, n \}, r \in \{ n+1, \ldots, 2m \} \), i.e. \( X \in \mathcal{N}_p \).

iii) The equality case of (2) holds for all unit tangent vectors at \( p \) if and only if

\[ \begin{cases} h^r_{ij} = 0, & i \neq j, r \in \{ n+1, \ldots, 2m \}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ij} = 0, & i \in \{ 1, \ldots, n \}, r \in \{ n+1, \ldots, 2m \}. \end{cases} \]

We distinguish two cases:

a) \( n \neq 2 \), then \( p \) is a totally geodesic point.

b) \( n = 2 \), it follows that \( p \) is a totally umbilical point.

The converse is trivial.

**Corollary 2.3.** Let \( M \) be a \( n \)-dimensional totally real submanifold in an \( m \)-dimensional generalized complex space form \( \tilde{M}(f_1, f_2) \). Then:

i) For each unit vector \( X \in T_pM \), we have

\[ \text{Ric} X \leq \frac{n^2}{4} \| H \|^2 + (n-1)f_1. \]  

(8)

ii) If \( H(p) = 0 \), then a unit tangent vector \( X \) at \( p \) satisfies the equality case of (8) if and only if \( X \in \mathcal{N}_p \).

iii) The equality case of (8) holds identically for all unit tangent vectors at \( p \) if and only if either \( p \) is a totally geodesic point or \( n = 2 \) and \( p \) is a totally umbilical point.
Theorem 2.4. Let $M$ be an $n$-dimensional $\theta$-slant submanifold in an $m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then the Ricci tensor $S$ satisfies

$$S \leq \left( \frac{n^2}{4} \|H\|^2 + (n - 1)f_1 + 3f_2 \cos^2 \theta \right) g$$

(9)

The equality case of (9) holds identically if and only if either $M$ is a totally geodesic submanifold or $n = 2$ and $M$ is a totally umbilical submanifold.

4. Squared Mean Curvature and Normalized Scalar Curvature

Theorem 2.5. Let $M$ be an $n$-dimensional $\theta$-slant submanifold in an $m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then we have

$$\|H\|^2 \geq \rho - f_1 - \frac{3f_2}{n - 1} \cos^2 \theta.$$  

(10)

equality holding at a point $p \in M^n$ if and only if $p$ is a totally umbilical point.

Proof. Choose an orthonormal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}$ at $p$ such that $e_{n+1}$ is parallel to the mean curvature vector and $e_1, \ldots, e_n$ diagonalize the shape operator $A_{n+1}$. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

(11)

$$A_r = (h^r_{ij}), \quad \sum_{i=1}^n h^r_{ii} = 0, \quad r = n + 2, \ldots, 2m.$$  

(12)

Then from the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau - n(n - 1)f_1 - 3nf_2 \cos^2 \theta + \sum_{i=1}^n a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h^r_{ij})^2.$$  

(13)

On the other hand, since

$$n \sum_{i=1}^n a_i^2 \geq \left( \sum_{i=1}^n a_i \right)^2.$$  

(14)

We get $\sum_{i=1}^n a_i^2 \geq n\|H\|^2$. Combining this with (13), we obtain

$$n(n - 1)\|H\|^2 \geq 2\tau - n(n - 1)f_1 - 3nf_2 \cos^2 \theta + \sum_{r=n+2}^{2m} \sum_{i,j=1}^n (h^r_{ij})^2.$$  

(15)
which implies inequality (10). If the equality sign of (10) holds at a point \( p \in M^n \), then from (14) and (15) we get \( A_r = 0, \ r = n + 2, \ldots, 2m \) and \( a_1 = \cdots = a_n \). Therefore, \( p \) is a totally umbilical point. The converse is trivial.

**Corollary 2.6.** \( \text{Let } M \text{ be a } n \)-dimensional totally real submanifold in an \( m \)-dimensional generalized complex space form \( \tilde{M}(f_1, f_2) \). Then:

\[
\|H\|^2 \geq \rho - f_1. \tag{16}
\]

equality holding at a point \( p \in M^n \) if and only if \( p \) is a totally umbilical point.

**References**


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