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SOME INEQUALITIES OF SLANT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

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Abstract. In this paper, we obtain an inequality about Ricci curvature and squared mean curvature of slant submanifolds in generalized complex space forms. We also obtain an inequality about the squared mean curvature and the normalized scalar curvature of slant submanifolds in generalized coplex space forms.

1. Introduction

To find simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold is one of the basic interests of study in the submanifold theory Scalar curvature and Ricci curvatureare among the main intrinsic invariants, while the squared mean curvature is the main extrinsic invariants. In [5], an inequality about the Ricci curvature and the squared mean curvature of a slant submanifold in complex space forms was obtained by K. Matsumoto, I. Mihai and Y. Tazawa. Also a sharp relationship between the normalized scalar curvature and the squared mean curvature of isometric in real-space-forms has been proofed by B. Y. Chen in [2].

On the other hand, the definition of the generalized complex space forms was introduced by F. Tricerri and L. Vanahecke in [4]. In the present paper, we study the slant submanifolds in generalized complex space forms, and we obtain an inequality about Ricci curvature and squared mean curvature of slant submanifolds in generalized complex space forms. Also we get an inequality about the squared mean curvature and the normalized scalar curvature of slant submanifolds in generalized complex space forms. These results generalize the corresponding results in [5] and [2].

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with an almost Hermitian structure $(J.\langle . \rangle)$. An almost Hermitian manifold becomes a *nearly Kähler manifold* [1] if $(\tilde{\nabla}_X J)X = 0$, and become a *Kähler manifold* if $\tilde{\nabla} J = 0$ for all $X \in T\tilde{M}$, where $\tilde{\nabla}$ is the levi-Civita

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connection of the Riemannian metric $\langle,\rangle$. An almost Hermitian manifold with J-invariant Riemannian curvature tensor $\tilde{R}$, that is,

$$\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W), \quad X, Y, Z, W \in T\tilde{M}$$

is called an *RK-manifolds* [4]. All nearly *Kähler manifolds* belong to the class of *RK-manifolds.*

The notion of constant type was first introduced by *A.* Gray for a nearly *Kähler manifold* ([1]). An almost Hermitian manifold $\tilde{M}$ is said to have (*pointwise*) *constant type* if for each $p \in \tilde{M}$ and for $X, Y, Z \in T_p\tilde{M}$ such that

$$\langle X, Y\rangle = \langle X, Z\rangle = \langle X, JY\rangle = \langle X, JZ\rangle = 0, \quad \langle Y, Y\rangle = 1 = \langle Z, Z\rangle$$

we have

$$\tilde{R}(X, Y.X, Y) - \tilde{R}(X, Y, JX, JY) = \tilde{R}(X, Z, X, Z) - \tilde{R}(X, Z, JX, JZ).$$

An RK-manifold $\tilde{M}$ has *pointwise* constant type if and only if there is a differentiable function α on $\tilde{M}$ satisfying [6]

$$\tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \alpha\{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2 - \langle X, Y\rangle^2\}$$

for all $X,\ Y \in T\tilde{M}$. Furthermore, $\tilde{M}$ has *global constant type* if α is constant. The fuction α is called the constant type of $\tilde{M}$.

An RK-*manifold* of constant holomorphic sectional curvature c and constant type α is denoted by $\tilde{M}(c, \alpha)$. For $\tilde{M}(c, \alpha)$ it is known that [6]

$$\begin{aligned} 4\tilde{R}(X, Y)Z = (c + 3\alpha)\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + (c - a)\{&\langle X, JZ\rangle JY - \langle Y, JZ\rangle JX \\ &+ 2\langle X.JY\rangle JZ\} \end{aligned}$$

for all $X, Y, Z \in T\tilde{M}$. If $c = \alpha$ then $\tilde{M}(c, \alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ (a *Kähler* manifold of constrant holomorphic sectional curvature c) belongs to the class of almost Hermitian manifolds $\tilde{M}(c, \alpha)$ (with the constant type zero).

An almost Hermitian manifold $\tilde{M}$ is called a generalized complex space form $\tilde{M}(f_1, f_2)$ [4] if its Riemannian curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X, Y)Z = f_1\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + f_2\{\langle X, JZ\rangle JY - \langle Y, JZ\rangle JX + 2\langle X, JY\rangle JZ\}$$

for all $X, Y, Z \in T\tilde{M}$, where f_1 and f_2 are smooth functions on $\tilde{M}$.

Let M be n-dimensional submanifold of an $2m$-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$, Let $p \in M$ and $\{e_1, \ldots, e_{2m}\}$ an orthonormal basis at p, such that $\{e_1, \ldots, e_n\}$ are tangent to M and $e_{n+1}, \ldots, e_{2m}$ are normal to M. We denote the sectional curvature of the plane section $\pi \in T_pM$ by $K(\pi)$. For an orthonormal basis

$e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature π and the normalized scalar curvature ρ are defined respectively by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \quad \rho = \frac{2\tau}{n(n-1)}.$$

Then the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))$$

for any vectors X, Y, Z, W tangent to M, where R is the Riemannian curvature tensor of M.

We denote by H the *mean curvature vector* at $p \in M$, i.e.

$$H(P) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, \ldots, e_{2m}\}$ is an orthonormal basis of the tangent space $T_p\tilde{M}(f_1, f_2)$, such that $\{e_1, \ldots, e_n\}$ are tangent to M.

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j = 1, \ldots, n; \; r = n+1, \ldots, 2m,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For any $p \in M$ and for any $X \in T_pM$, we put $JX = PX + FX$, where $PX \in T_pM$, $FX \in T_p^{\perp}M$.

We put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

A submanifold M of an almost Hermitian manifold $\tilde{M}$ is said to be a *slant submanifold* if for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between JX and the tangent space T_pM is costant $(= \theta)$. M is called a *totally real submanifold* if the almost complex structure J of $\tilde{M}$ carries each tangent space of M^n into its corresponding normal space (i.e. $\theta = \frac{\pi}{2}$).

If M is a θ-*slant* submanifold in a generalized complex space form $\tilde{M}(f_1, f_2)$, then

$$\|P\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g^2(Pe_i, e_j) = \sum_{i=1}^{n} \|Pe_i\|^2 = n \cos^2 \theta.$$

We recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point p is defined by

$$\mathcal{N}_p = \{X \in T_pM \mid h(X, Y) = 0, \text{ for all } Y \in T_pM\}.$$

3. Ricci Tensor and Squared Mean Curvature

Let M be a submanifold in a generalized complex space form, then the Gauss equation becomes [7]

$$\begin{aligned} R(X,Y,Z,W) = & f_1\{\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle\} \\ & + f_2\{\langle X,PZ\rangle\langle PY,W\rangle - \langle Y,PZ\rangle\langle PX,W\rangle + 2\langle X,PY\rangle\langle PZ,W\rangle\} \\ & + \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(Y,W)\rangle \end{aligned}$$

for all $X,Y,Z,W \in TM$. Thus, we are able to state the following lemma.

Lemma 2.1. *For an n-dimensional submanifold in a generalized complex space form $\tilde{M}(f_1,f_2)$, the scalar curvature and the squared mean curvature satisfy*

$$2\tau = n(n-1)f_1 + 3f_2\|P\|^2 + n^2\|H\|^2 - \|h\|^2. \tag{1}$$

First we will prove the following basic inequality. It generalizes Theory 2.1 *in* [5].

Theorem 2.2. *Let M be an n-dimensional θ-slant submanifold in an m-dimensional generalized complex space form $\tilde{M}(f_1,f_2)$. Then:*
i) *For each unit vector $X \in T_pM$, we have*

$$Ric(X) \le \frac{n^2}{4}\|H\|^2 + (n-1)f_1 + 3f_2\cos^2\theta. \tag{2}$$

ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of* (2.1) *if and only if $X \in \mathcal{N}_p$.*
iii) *The equality case of* (2) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Let $X \in T_pM$ be a unit tangent vector x at p. We choose an orthogonal basis $e_1,\dots,e_n,e_{n+1},\dots,e_{2m}$ such that $e_1,\dots,e_n$ are tangent to M at p, with $e_1 = X$. Then from (1), we have

$$n^2\|h\|^2 = 2\tau + \|h\|^2 - n(n-1)f_1 - 3nf_2\cos^2\theta. \tag{3}$$

Form (3) we get

$$\begin{aligned} n^2\|H\|^2 = & 2\tau + \sum_{r=n+1}^{2m}\Big[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r) + 2\sum_{i<j}(h_{ij}^2)^2\Big] \\ & -2\sum_{r=n+1}^{2m}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - [n(n-1)f_1 + 3nf_2\cos^2\theta] \\ = & 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m}[(h_{11}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2] \\ & +2\sum_{r=n+1}^{2m}\sum_{i<j}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - [n(n-1)f_1 + 3nf_2\cos^2\theta]. \end{aligned} \tag{4}$$

From the equation of Gauss, we have

$$K_{ij} = \sum_{r=n+1}^{2m} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + f_1 + 3f_2 g^2(Pe_i, e_j). \tag{5}$$

and consequently

$$\sum_{2\leq i<j\leq n} K_{ij} = \sum_{r=n+1}^{2m} \sum_{2\leq i<j\leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2} f_1 + \frac{3}{2} n f_2 \cos^2\theta - 3f_2\cos^2\theta. \tag{6}$$

Substituting (6) in (4), we have

$$\begin{aligned} n^2\|H\|^2 \geq 2\tau + \frac{1}{2}n^2\|H\|^2 + 2\sum_{r=n+1}^{2m}\sum_{j=2}^{n}(h_{1j}^r)^2 - 2\sum_{2\leq i<j\leq n} K_{ij} \\ +[(n-1)(n-2) - n(n-1)]f_1 - 6f_2\cos^2\theta. \end{aligned} \tag{7}$$

From inequality (7) we could get (2).

ii) Assume $H(p) = 0$, equality holds in (2) if and only if

$$\begin{cases} h_{12}^r = \cdots = h_{1n}^r = 0. \\ h_{11}^r = h_{22}^r + \cdots + h_{nn}^r, \ r \in \{n+1, \ldots, 2m\}. \end{cases}$$

Then $h_{1j}^r = 0 \ \forall j \in \{1, \ldots, n\}$, $r \in \{n+1, \ldots, 2m\}$, i.e. $X \in \mathcal{N}_p$.

iii) The equality case of (2) holds for all unit tangent vectors at p if and only if

$$\begin{cases} h_{ij}^r = 0, \ i \neq j, \ r \in \{n+1, \ldots, 2m\}, \\ h_{11}^r + \cdots + h_{nn}^r - 2h_{ij}^r = 0, \ i \in \{1, \ldots, n\}, \ r \in \{n+1, \ldots, 2m\}. \end{cases}$$

We distinguish two cases:

a) $n \neq 2$, then p is a totally geodesic point.

b) $n = 2$, it follows that p is a totally umbilical point.

The converse is trivial.

Corollary 2.3. *Let M be a n-dimensional totally real submanifold in an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then:*

i) *For each unit vector $X \in T_pM$, we have*

$$RicX \leq \frac{n^2}{4}\|H\|^2 + (n-1)f_1. \tag{8}$$

ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case of* (8) *if and only if $X \in \mathcal{N}_p$.*

iii) *The equality case of* (8) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Theorem 2.4. *Let M be an n-dimensional θ-slant submanifold in an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then the Ricci tensor S satisfies*

$$S \leq \left(\frac{n^2}{4} \|H\|^2 + (n-1) f_1 + 3 f_2 \cos^2 \theta \right) g \tag{9}$$

The equality case of (9) *holds identically if and only if either M is a totally geodesic submanifold or $n = 2$ and M is a totally umbilical submanifold.*

4. Squared Mean Curvature and Normalized Scalar Curvature

Theorem 2.5. *Let M be an n-dimensional θ-slant submanifold in an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then we have*

$$\|H\|^2 \geq \rho - f_1 - \frac{3 f_2}{n-1} \cos^2 \theta. \tag{10}$$

equality holding at a point $p \in M^n$ if and only if p is a totally umbilical point.

Proof. Choose an orthonomal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}$ at p such that e_{n+1} is parallel to the mean curvature vector and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}. \tag{11}$$

$$A_r = (h_{ij}^r), \quad \sum_{i=1}^{n} h_{ii}^r = 0, \quad r = n+2, \ldots, 2m. \tag{12}$$

Then from the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau - n(n-1) f_1 - 3 n f_2 \cos^2 \theta + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2. \tag{13}$$

On the other hand, since

$$n \sum_{i=1}^{n} a_i^2 \geq \Big(\sum_{i=1}^{n} a_i \Big)^2. \tag{14}$$

We get $\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2$. Combining this with (13), we obtain

$$n(n-1) \|H\|^2 \geq 2\tau - n(n-1) f_1 - 3 n f_2 \cos^2 \theta + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2. \tag{15}$$

which implies inequality (10). If the equality sign of (10) holds at a point $p \in M^n$, then from (14) and (15) we get $A_r = 0$, $r = n+2, \ldots, 2m$ and $a_1 = \cdots = a_n$. Therefore, p is a totally umbilical point. The converse is trivial.

Corollary 2.6. *Let M be a n-dimensional totally real submanifold in an m-dimensional generalized complex space form $\tilde{M}(f_1, f_2)$. Then:*

$$\|H\|^2 \geq \rho - f_1. \tag{16}$$

equality holding at a point $p \in M^n$ if and only if p is a totally umbilical point.

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