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OSTROWSKI TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper, we obtain Ostrowski type inequalities for convex functions.

1. Introduction

A function $f : I \to \mathbb{R}$ is said to be convex function on I if the inequality

$$f\left(\alpha x + (1-\alpha)\, y\right) \le \alpha f(x) + (1-\alpha)\, f(y)$$

holds for all $x, y \in I$ and $\alpha \in [0,1]$.

Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$ where $a, b \in I$ with $a < b$. If $\left|f'\right| \le M$, then the following inequality holds (See [1])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \le \frac{M}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right]. \tag{1.1}$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve and extend the inequality (1.1) see [1]-[14].

2. Inequalities for Convex Functions

We use the following Lemma to obtain our main results.

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval and $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$. If $f' \in L_1[a,b]$ for $\lambda \in [0,1]$ and $x \in [a,b]$, the following equality holds*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(u)\,du - \frac{(b-x)\left[(1-\lambda)\, f(x) + \lambda f(b)\right] + (x-a)\left[(1-\lambda)\, f(x) + \lambda f(a)\right]}{b-a} \\ &\quad = (b-a)\int_0^1 h(t,\lambda)\, f'(ta + (1-t)b)\,dt \end{aligned}$$

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where

$$h(t,\lambda)=\begin{cases} t-\lambda\left(\frac{b-x}{b-a}\right), & t\in\left[0,\frac{b-x}{b-a}\right) \\ t-1+\lambda\left(\frac{x-a}{b-a}\right), & t\in\left(\frac{b-x}{b-a},1\right]. \end{cases}$$

Proof. We note that

$$\begin{aligned} K &= \int_0^1 h(t,\lambda) f'(ta+(1-t)b)\,dt \\ &= \int_0^{\frac{b-x}{b-a}} \left[t-\lambda\left(\frac{b-x}{b-a}\right)\right] f'(ta+(1-t)b)\,dt + \int_{\frac{b-x}{b-a}}^1 \left[t-1+\lambda\left(\frac{x-a}{b-a}\right)\right] f'(ta+(1-t)b)\,dt. \end{aligned}$$

Integrating by parts we have

$$\begin{aligned} K &= \left[t-\lambda\left(\frac{b-x}{b-a}\right)\right] \frac{f(ta+(1-t)b)}{a-b}\Bigg|_0^{\frac{b-x}{b-a}} - \int_0^{\frac{b-x}{b-a}} \frac{f(ta+(1-t)b)}{a-b}\,dt \\ &\quad + \left[t-1+\lambda\left(\frac{x-a}{b-a}\right)\right] \frac{f(ta+(1-t)b)}{a-b}\Bigg|_{\frac{b-x}{b-a}}^1 - \int_{\frac{b-x}{b-a}}^1 \frac{f(ta+(1-t)b)}{a-b}\,dt \\ &= \frac{1}{(b-a)^2}\int_a^b f(u)\,du - \frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{(b-a)^2}. \end{aligned}$$

If we multiply the resulting equality with $(b-a)$, we complete the proof. □

Theorem 1. *Let $I\subset\mathbb{R}$ be an open interval and $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function where $a,b\in I$ with $a<b$. If $|f'|^q$ is convex function for $\lambda\in[0,1]$, $x\in[a,b]$ and $q\in[1,\infty)$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(u)\,du - \frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{b-a}\right| \\ &\le (b-a)\left(\frac{(2\lambda^2-2\lambda+1)}{2}\right)^{\frac{(q-1)}{q}} \Bigg\{\Bigg[\left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{b-x}{b-a}\right)^{2q+1}|f'(a)|^q \\ &\quad + \left(\frac{[6\lambda^2-6\lambda+3]-[2\lambda^3-3\lambda+2]\left(\frac{b-x}{b-a}\right)}{6}\right)\left(\frac{b-x}{b-a}\right)^{2q}|f'(b)|^q\Bigg]^{\frac{1}{q}} \\ &\quad + \Bigg[\left(\frac{[6\lambda^2-6\lambda+3]-[2\lambda^3-3\lambda+2]\left(\frac{x-a}{b-a}\right)}{6}\right)\left(\frac{x-a}{b-a}\right)^{2q}|f'(a)|^q \\ &\quad + \left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{x-a}{b-a}\right)^{2q+1}|f'(b)|^q\Bigg]^{\frac{1}{q}}\Bigg\}. \end{aligned}$$

Proof. Suppose that $q=1$. From Lemma 1 and using the properties of modulus, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - \frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{b-a}\right|$$

$$\leq (b-a)\left\{\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|\left|f'(ta+(1-t)b)\right|dt\right.$$
$$\left.+\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|\left|f'(ta+(1-t)b)\right|dt\right\}.$$

Since $|f'|$ is convex function on $[a,b]$, we can write

$$\left|f'(ta+(1-t)b)\right|\leq t\left|f'(a)\right|+(1-t)\left|f'(b)\right|.$$

Hence,

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{b-a}\right|$$
$$\leq (b-a)\left\{\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right.$$
$$\left.+\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|\left[t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right\}$$
$$=(b-a)\left\{\left|f'(a)\right|\left[\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|tdt+\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|tdt\right]\right.$$
$$\left.+\left|f'(b)\right|\left[\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|(1-t)\,dt+\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|(1-t)\,dt\right]\right\}. \tag{2.1}$$

To complete the proof for this case we used the fact that

$$\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|tdt=\left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{b-x}{b-a}\right)^3, \tag{2.2}$$
$$\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|tdt=\left(\frac{\left[6\lambda^2-6\lambda+3\right]-\left[2\lambda^3-3\lambda+2\right]\left(\frac{x-a}{b-a}\right)}{6}\right)\left(\frac{x-a}{b-a}\right)^2, \tag{2.3}$$
$$\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|(1-t)\,dt=\left(\frac{\left[6\lambda^2-6\lambda+3\right]-\left[2\lambda^3-3\lambda+2\right]\left(\frac{b-x}{b-a}\right)}{6}\right)\left(\frac{b-x}{b-a}\right)^2 \tag{2.4}$$

and

$$\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|(1-t)\,dt=\left(\frac{2\lambda^3-3\lambda+2}{6}\right)\left(\frac{x-a}{b-a}\right)^3 \tag{2.5}$$

in (2.1).

Now, suppose that $q>1$. From Lemma 1, using Hölder inequality and convexity of $|f'|^q$, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{b-a}\right|$$

$$
\begin{aligned}
&\le (b-a)\left\{\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|^{\frac{q-1}{q}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|^{\frac{1}{q}}\left|f'(ta+(1-t)b)\right|dt\right.\\
&\quad\left.+\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|^{\frac{q-1}{q}}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|^{\frac{1}{q}}\left|f'(ta+(1-t)b)\right|dt\right\}\\
&\le (b-a)\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|dt\right)^{\frac{q-1}{q}}\right.\\
&\quad\times\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|dt\right)^{\frac{q-1}{q}}\\
&\quad\left.\times\left(\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&\le (b-a)\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|dt\right)^{\frac{q-1}{q}}\right.\\
&\quad\times\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|dt\right)^{\frac{q-1}{q}}\\
&\quad\left.\times\left(\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

If we use the inequalities in (2.2)-(2.5) and the following two inequalities in the last inequality, we obtain the required result.

We note that

$$\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|dt=\left(\frac{2\lambda^2-2\lambda+1}{2}\right)\left(\frac{b-x}{b-a}\right)^2$$

and

$$\int_{\frac{b-x}{b-a}}^{1}\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|dt=\left(\frac{2\lambda^2-2\lambda+1}{2}\right)\left(\frac{x-a}{b-a}\right)^2. \qquad \square$$

Theorem 2. *Let $I\subset\mathbb{R}$ be an open interval and $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function where $a,b\in I$ with $a<b$. If $\left|f'\right|^q$ is convex function for $\lambda\in[0,1]$, $x\in[a,b]$ and $q\in(1,\infty)$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{b-a}\right|$$

$$\leq (b-a)\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{b-x}{b-a}\right)^{\frac{p+1}{p}}\left(\frac{\left(\frac{b-x}{b-a}\right)^2}{2}\left|f'(a)\right|^q+\left(\frac{1}{2}-\frac{\left(\frac{x-a}{b-a}\right)^2}{2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{x-a}{b-a}\right)^{\frac{p+1}{p}}\left(\left(\frac{1}{2}-\frac{\left(\frac{b-x}{b-a}\right)^2}{2}\right)\left|f'(a)\right|^q+\frac{\left(\frac{x-a}{b-a}\right)^2}{2}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Proof. From Lemma 1, using the properties of modulus and Hölder inequality, we have

$$\left|\frac{1}{b-a}\int_a^b f(u)du-\frac{(b-x)\left[(1-\lambda)f(x)+\lambda f(b)\right]+(x-a)\left[(1-\lambda)f(x)+\lambda f(a)\right]}{b-a}\right| \tag{2.6}$$
$$\leq (b-a)\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f'\right|^q$ is convex function we can write

$$\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt \leq \frac{\left(\frac{b-x}{b-a}\right)^2}{2}\left|f'(a)\right|^q+\left(\frac{1}{2}-\frac{\left(\frac{x-a}{b-a}\right)^2}{2}\right)\left|f'(b)\right|^q \tag{2.7}$$

and

$$\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt = \left(\frac{1}{2}-\frac{\left(\frac{b-x}{b-a}\right)^2}{2}\right)\left|f'(a)\right|^q+\frac{\left(\frac{x-a}{b-a}\right)^2}{2}\left|f'(b)\right|^q. \tag{2.8}$$

By a simple calculation, we also have

$$\int_0^{\frac{b-x}{b-a}}\left|t-\lambda\left(\frac{b-x}{b-a}\right)\right|^p dt = \left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)\left(\frac{b-x}{b-a}\right)^{p+1} \tag{2.9}$$

and

$$\int_{\frac{b-x}{b-a}}^1\left|t-1+\lambda\left(\frac{x-a}{b-a}\right)\right|^p dt = \left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}\right)\left(\frac{x-a}{b-a}\right)^{p+1}. \tag{2.10}$$

By using the inequalities in (2.7)-(2.10) in the inequality (2.6), we obtained the required result. □

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