OSCILLATORY BEHAVIOR OF SOLUTIONS OF CERTAIN THIRD ORDER MIXED NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. The objective of this paper is to study the oscillatory and asymptotic properties of third order mixed neutral differential equation of the form

$$(a(t)[x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2)]'')'+q(t)x^{\alpha}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2)=0$$

where $a(t), b(t), c(t), q(t)$ and $p(t)$ are positive continuous functions, α and β are ratios of odd positive integers, τ_1, τ_2, σ_1 and σ_2 are positive constants. We establish some sufficient conditions which ensure that all solutions are either oscillatory or converge to zero. Some examples are provided to illustrate the main results.

1. Introduction

In this paper, we are concerned with the following third order mixed neutral type differential equation of the form

$$(a(t)[x(t)+b(t)x(t-\tau_1)+c(t)x(t+\tau_2)]'')'+q(t)x^{\alpha}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2)=0, \quad (1.1)$$

for $t \geq t_0$. Throughout this paper, we assume that the following hypotheses hold.

(H$_1$) $a(t)$ is a positive nondecreasing continuous function for all $t \geq t_0$ with $\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty$;

(H$_2$) $b(t), c(t) \in C([t_0,\infty),(0,\infty))$ and there exist b and c such that $b(t) \leq b, c(t) \leq c$ with $b+c<1$;

(H$_3$) $p(t), q(t) \in C([t_0,\infty),(0,\infty))$;

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(H$_4$) τ_1, τ_2, σ_1 and σ_2 are nonnegative constants and α and β are ratios of odd positive integers.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of equation (1.1), we mean a real continuous function $x(t)$ defined for all $t \geq t_0 - \theta$ and satisfying the equation (1.1) for all $t \geq t_0$. A solution of equation (1.1) is called oscillatory if it has large zeros on $[t_0, \infty)$, otherwise it is called nonoscillatory.

Recently there has been a great interest in studying the oscillatory and asymptotic behavior of differential equations, see for example [1–24] and the references cited therein. Especially the equation (1.1) with $c(t) \equiv 0$ and $p(t) \equiv 0$ have been the subject of intensive research. In [1, 4–6, 9–11, 16, 21, 24], the authors studied the oscillatory behavior of solutions of equation (1.1) when $b(t) \equiv 0$, $c(t) \equiv 0$ and $p(t) \equiv 0$. In [7, 8, 12, 13, 17–20, 22], the authors studied the oscillatory behavior of solutions of equation (1.1) when $c(t) \equiv 0$ and $p(t) \equiv 0$. In [2, 14, 15, 23], the authors discussed the oscillatory behavior of all solutions of equation (1.1) when $\alpha = \beta = 1$.

It is interesting to study the equation (1.1) under the conditions $\alpha = \beta$ and $\alpha \neq \beta$. To the best of our knowledge, there are no results regarding the oscillation of equation (1.1) under the assumption $\alpha \neq \beta$. So the purpose of this paper is to present some new oscillatory and asymptotic criteria for equation (1.1). In Section 2, we present criteria for equation (1.1) to be oscillatory or for all its nonoscillatory solutions tend to zero as $t \to \infty$. Examples are provided in Section 3 to illustrate the results presented in Section 2.

2. Oscillatory Results

In this section, we present some new oscillation criteria for equation (1.1). For the sake of convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all large t.

We begin with the following lemmas which are crucial in the proof of the main results. For simplicity, we use the following notations, without further mention:

$$\begin{aligned} z(t) &= x(t) + b(t)x(t-\tau_1) + c(t)x(t+\tau_2), \\ Q(t) &= \min\{q(t), q(t-\tau_1), q(t+\tau_2)\}, \quad P(t) = \min\{p(t), p(t-\tau_1), p(t+\tau_2)\}, \\ R(t) &= Q(t) + P(t), \\ \eta(t) &= \left(\frac{d}{4}\right)^{\beta-1} \frac{k(t-\sigma)^{\beta}}{2^{\beta}} R(t) \text{ for some } k \in (0,1),\ \sigma = \max(\sigma_1, \sigma_2) \text{ and } d > 0. \end{aligned}$$

Lemma 2.1. *Assume $A \geq 0, B \geq 0$. If $\delta \geq 1$, then*

$$(A+B)^{\delta} \leq 2^{\delta-1}(A^{\delta} + B^{\delta}).$$

If $0 < \delta \leq 1$, then $(A+B)^{\delta} \leq A^{\delta} + B^{\delta}$.

Proof. Proof can be found in [22]. ∎

Lemma 2.2. *Let $x(t)$ be a positive solution of equation (1.1). Then there are only two cases for $z(t)$ for all sufficiently large $t \geq t_1$.*

(I) $z(t) > 0,\ \ z'(t) > 0,\ \ z''(t) > 0,\ \ (a(t)\ z''(t))' \leq 0;$
(II) $z(t) > 0,\ \ z'(t) < 0,\ \ z''(t) > 0,\ \ (a(t)\ z''(t))' \leq 0.$

Proof. Let $x(t)$ be a positive solution of equation (1.1). Then there exists a $t_1 \geq t_0$ such that $x(t) > 0,\ x(t-\sigma_1) > 0$ and $x(t-\tau_1) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for all $t \geq t_1$. It follows from equation (1.1) that

$$(a(t)\ z''(t))' = -q(t)x^{\alpha}(t-\sigma_1) - p(t)x^{\beta}(t+\sigma_2) < 0, \quad t \geq t_1. \tag{2.1}$$

Hence $a(t)z''(t)$ is strictly decreasing for all $t \geq t_1$. We claim that $z''(t) > 0$ for all $t \geq t_1$. If not, then there is a $t_2 \geq t_1$ and $M < 0$ such that

$$a(t)\ z''(t) \leq a(t_2)\ z''(t_2) \leq M \text{ for all } t \geq t_2.$$

Integrating the last inequality from t_2 to t, we have

$$z'(t) \le z'(t_2) + M \int_{t_2}^{t} \frac{1}{a(s)} ds.$$

Letting $t \to \infty$, and using (H_1) we see that $z'(t) \to -\infty$. Thus there exists a $t_3 \ge t_2$ such that $z'(t) < 0$ for all $t \ge t_3$. This implies that $z(t) \to -\infty$ as $t \to \infty$, a contradiction. Hence $z''(t) > 0$ for all $t \ge t_1$. This completes the proof. ■

Lemma 2.3. *Let $z(t) > 0, z'(t) > 0, z''(t) > 0$ and $z'''(t) \le 0$ for all $t \ge t_0$. Then for some $k \in (0,1)$ and for some t_1*

$$\frac{z(t)}{z'(t)} \ge \frac{(t-t_0)}{2} \ge \frac{kt}{2} \text{ for } t \ge t_1 \ge t_0. \tag{2.2}$$

Proof. Since $z''(t)$ is nonincreasing and

$$z'(t) = z'(t_0) + \int_{t_0}^{t} z''(s) ds,$$

we have

$$z'(t) \ge (t-t_0) z''(t).$$

Integrating the last inequality from t_0 to t, we have

$$z(t) \ge z(t_0) + (t-t_0) z'(t) - z(t) + z(t_0)$$

or

$$z(t) \ge \left(\frac{t-t_0}{2}\right) z'(t) \ge \frac{kt}{2} z'(t) \text{ for some } k \in (0,1).$$

The proof is now complete. ■

Lemma 2.4. *Let $x(t)$ be a positive solution of equation (1.1), $\alpha = \beta \ge 1$ and the corresponding $z(t)$ satisfies Lemma 2.2 (II). If*

$$\int_{t_0}^{\infty} \left(\int_{t}^{\infty} \left(\frac{1}{a(s)} \int_{s}^{\infty} (q(u)+p(u)) du \right) ds \right) dt = \infty \tag{2.3}$$

holds, then $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a positive solution of equation (1.1). Then $z(t) > 0$ and $z'(t) < 0$, we have $\lim_{t\to\infty} z(t) = l \geq 0$ exists. We shall prove that $l = 0$. Assume that $l > 0$. Then for any $\epsilon > 0$, we have $l + \epsilon > z(t)$ eventually. Choose

$$0 < \epsilon < \frac{l(1-b-c)}{b+c}.$$

It is easy to verify that

$$\begin{aligned} x(t) &= z(t) - b(t)x(t-\tau_1) - c(t)x(t+\tau_2) \\ &> l - (b+c)z(t-\tau_1) \\ &> l - (b+c)(l+\epsilon) \\ &= k(l+\epsilon) > kz(t), \end{aligned}$$

where $k = \dfrac{l-(b+c)(l+\epsilon)}{l+\epsilon} > 0$. Using the above inequality, we obtain from (2.1)

$$\begin{aligned} (a(t)\ z''(t))' &\leq -q(t)k^{\beta}z^{\beta}(t-\sigma_1) - p(t)k^{\beta}z^{\beta}(t+\sigma_2) \\ &\leq -k^{\beta}(q(t)+p(t))z^{\beta}(t-\sigma_1). \end{aligned}$$

Integrating the above inequality from t to ∞ and using $z(t) > l$, we obtain

$$z''(t) \geq (kl)^{\beta}\left[\frac{1}{a(t)}\int_t^{\infty}(p(s)+q(s))ds\right].$$

Integrating again from t to ∞, we have

$$-z'(t) \geq (kl)^{\beta}\int_t^{\infty}\frac{1}{a(s)}\left(\int_s^{\infty}(p(t)+q(t))dt\right)ds.$$

Integrating from t_1 to ∞, we obtain

$$z(t_1) \geq (kl)^{\beta}\int_{t=t_1}^{\infty}\left(\int_t^{\infty}\left(\frac{1}{a(s)}\int_s^{\infty}(p(u)+q(u))du\right)ds\right)dt.$$

This contradicts (2.3). Hence $l = 0$, moreover the inequality $0 \leq x(t) \leq z(t)$ implies that $\lim_{t\to\infty} x(t) = 0$ and the proof is complete. ∎

Next, we establish some oscillation results which ensure that every solution of equation (1.1) either oscillates or converges to zero.

Theorem 2.5. *Assume that condition* (2.3) *holds,* $\sigma_1 \geq \tau_1$, *and* $\alpha = \beta \geq 1$. *If there exists a positive real valued function* $\rho(t)$ *and* $t_1 > 0$ *with*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\rho(s)\eta(s) - \frac{\left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right)}{4} \frac{a(s-\sigma_1)(\rho'(s))^2}{\rho(s)} \right] ds = \infty \tag{2.4}$$

holds, then every solution $x(t)$ *of equation* (1.1) *either oscillates or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\sigma_1) > 0$ and $x(t-\tau_1) > 0$ for all $t \geq t_1$. Then we have $z(t) > 0$ and (2.1) for all $t \geq t_1$. From the equation (1.1), we have

$$\begin{aligned}
&(a(t)\ z''(t))' + q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) + b^\beta(a(t-\tau_1)z''(t-\tau_1))' \\
&+ b^\beta q(t-\tau_1)x^\beta(t-\tau_1-\sigma_1) + b^\beta p(t-\tau_1)x^\beta(t-\tau_1+\sigma_2) + \frac{c^\beta}{2^{\beta-1}}(a(t+\tau_2)z''(t+\tau_2))' \\
&+ \frac{c^\beta}{2^{\beta-1}} q(t+\tau_2)x^\beta(t+\tau_2-\sigma_1) + \frac{c^\beta}{2^{\beta-1}} p(t+\tau_2)x^\beta(t+\tau_2+\sigma_2) = 0
\end{aligned} \tag{2.5}$$

That is,

$$\begin{aligned}
&(a(t)\ z''(t))' + b^\beta(a(t-\tau_1)z''(t-\tau_1))' + \frac{c^\beta}{2^{\beta-1}}(a(t+\tau_2)z''(t+\tau_2))' \\
&+ Q(t)\left[x^\beta(t-\sigma_1) + b^\beta x^\beta(t-\tau_1-\sigma_1) + \frac{c^\beta}{2^{\beta-1}}x^\beta(t+\tau_2-\sigma_1)\right] \\
&+ P(t)\left[x^\beta(t+\sigma_2) + b^\beta x^\beta(t-\tau_1+\sigma_2) + \frac{c^\beta}{2^{\beta-1}}x^\beta(t+\tau_2+\sigma_2)\right] \leq 0.
\end{aligned}$$

Applying Lemma 2.2 twice, the above inequality becomes

$$\begin{aligned}
&(a(t)\ z''(t))' + b^\beta(a(t-\tau_1)z''(t-\tau_1))' + \frac{c^\beta}{2^{\beta-1}}(a(t+\tau_2)z''(t+\tau_2))' \\
&+ \frac{Q(t)}{4^{\beta-1}}z^\beta(t-\sigma_1) + \frac{P(t)}{4^{\beta-1}}z^\beta(t+\sigma_2) \leq 0.
\end{aligned} \tag{2.6}$$

By Lemma 2.2, there are two cases for $z(t)$. First let us assume that Lemma 2.2(I) holds for all $t \geq t_1 \geq t_0$. Then $z'(t) > 0$ implies $z(t+\sigma_2) > z(t-\sigma_1)$. Thus from (2.6), we obtain

$$(a(t)\, z''(t))' + b^\beta (a(t-\tau_1) z''(t-\tau_1))' + \frac{c^\beta}{2^{\beta-1}} (a(t+\tau_2) z''(t+\tau_2))' + \frac{R(t)}{4^{\beta-1}} z^\beta (t-\sigma_1) \leq 0. \tag{2.7}$$

Define a function $w_1(t)$ by

$$w_1(t) = \frac{\rho(t) a(t) z''(t)}{z'(t-\sigma_1)} \quad \text{for all} \;\; t \geq t_1. \tag{2.8}$$

Then $w_1(t) > 0$ for all $t \geq t_1$. Differentiating (2.8), we obtain

$$w_1'(t) = \rho'(t) \frac{a(t) z''(t)}{z'(t-\sigma_1)} + \rho(t) \frac{(a(t) z''(t))'}{z'(t-\sigma_1)} - \rho(t) \frac{a(t) z''(t)}{(z'(t-\sigma_1))^2} z''(t-\sigma_1).$$

Since $a(t)z''(t)$ is strictly decreasing, we have $a(t-\sigma_1) z''(t-\sigma_1) \geq a(t)\; z''(t)$. Therefore,

$$\begin{aligned} w_1'(t) &\leq \rho'(t) \frac{a(t) z''(t)}{z'(t-\sigma_1)} + \rho(t) \frac{(a(t) z''(t))'}{z'(t-\sigma_1)} - \rho(t) \frac{a(t) z''(t)}{(z'(t-\sigma_1))^2} \frac{a(t) z''(t)}{a(t-\sigma_1)} \\ &\leq \rho'(t) \frac{a(t) z''(t)}{z'(t-\sigma_1)} + \rho(t) \frac{(a(t) z''(t))'}{z'(t-\sigma_1)} - \rho(t) \frac{(a(t) z''(t))^2}{(z'(t-\sigma_1))^2\; a(t-\sigma_1)} \\ w_1'(t) &\leq \frac{\rho'(t) w_1(t)}{\rho(t)} + \frac{\rho(t) (a(t) z''(t))'}{z'(t-\sigma_1)} - \frac{w_1^2(t)}{\rho(t) a(t-\sigma_1)}. \end{aligned} \tag{2.9}$$

Next, we define a function $w_2(t)$ by

$$w_2(t) = \frac{\rho(t) a(t-\tau_1) z''(t-\tau_1)}{z'(t-\sigma_1)} \quad \text{for all} \;\; t \geq t_1. \tag{2.10}$$

Then $w_2(t) > 0$ for all $t \geq t_1$. Differentiating (2.10), and similar to (2.9) we have

$$w_2'(t) \leq \frac{\rho'(t) w_2(t)}{\rho(t)} + \frac{\rho(t)(a(t-\tau_1) z''(t-\tau_1))'}{z'(t-\sigma_1)} - \frac{w_2^2(t)}{\rho(t) a(t-\sigma_1)}. \tag{2.11}$$

Define a function $w_3(t)$ by

$$w_3(t) = \frac{\rho(t) a(t+\tau_2) z''(t+\tau_2)}{z'(t-\sigma_1)} \quad \text{for all} \;\; t \geq t_1. \tag{2.12}$$

Then $w_3(t) > 0$ for all $t > t_1$. Differentiating (2.12),and similar to (2.9) we have

$$w_3'(t) \leq \frac{\rho'(t) w_3(t)}{\rho(t)} + \frac{\rho(t)(a(t+\tau_2) z''(t+\tau_2))'}{z'(t-\sigma_1)} - \frac{w_3^2(t)}{\rho(t) a(t-\sigma_1)}. \tag{2.13}$$

From (2.9), (2.11) and (2.13), we have

$$\begin{aligned}
w_1'(t)+b^\beta w_2'(t)+\frac{c^\beta}{2^{\beta-1}}w_3'(t) \leq\ & \frac{\rho'(t)w_1(t)}{\rho(t)}+\frac{\rho(t)(a(t)z''(t))'}{z'(t-\sigma_1)}-\frac{w_1^2(t)}{\rho(t)a(t-\sigma_1)}\\
&+b^\beta\left[\frac{\rho'(t)w_2(t)}{\rho(t)}+\frac{\rho(t)(a(t-\tau_1)z''(t-\tau_1))'}{z'(t-\sigma_1)}\right.\\
&\left.-\frac{w_2^2(t)}{\rho(t)a(t-\sigma_1)}\right]+\frac{c^\beta}{2^{\beta-1}}\left[\frac{\rho'(t)w_3(t)}{\rho(t)}\right.\\
&\left.+\frac{\rho(t)(a(t+\tau_2)z''(t+\tau_2))'}{z'(t-\sigma_1)}-\frac{w_3^2(t)}{\rho(t)a(t-\sigma_1)}\right]\\
=\ & \frac{\rho(t)}{z'(t-\sigma_1)}\left[(a(t)\ z''(t))'+b^\beta(a(t-\tau_1)z''(t-\tau_1))'\right.\\
&\left.+\frac{c^\beta}{2^{\beta-1}}(a(t+\tau_2)z''(t+\tau_2))'\right]+\left[\frac{\rho'(t)w_1(t)}{\rho(t)}\right.\\
&\left.-\frac{w_1^2(t)}{\rho(t)a(t-\sigma_1)}\right]+b^\beta\left[\frac{\rho'(t)w_2(t)}{\rho(t)}-\frac{w_2^2(t)}{\rho(t)a(t-\sigma_1)}\right]\\
&+\frac{c^\beta}{2^{\beta-1}}\left[\frac{\rho'(t)w_3(t)}{\rho(t)}-\frac{w_3^2(t)}{\rho(t)a(t-\sigma_1)}\right]\Bigg]\\
\leq\ & \frac{\rho(t)}{z'(t-\sigma_1)}\left[-k^\beta(q(t)+p(t))z^\beta(t-\sigma_1)-b^\beta k^\beta(q(t-\tau_1)\right.\\
&+p(t-\tau_1))z^\beta(t-\tau_1-\sigma_1)-\frac{c^\beta}{2^{\beta-1}}k^\beta(q(t+\tau_2)\\
&+p(t+\tau_2))z^\beta(t+\tau_2-\sigma_1)+\left[\frac{\rho'(t)w_1(t)}{\rho(t)}\right.\\
&\left.-\frac{w_1^2(t)}{\rho(t)a(t-\sigma_1)}\right]+b^\beta\left[\frac{\rho'(t)w_2(t)}{\rho(t)}-\frac{w_2^2(t)}{\rho(t)a(t-\sigma_1)}\right]\\
&+\frac{c^\beta}{2^{\beta-1}}\left[\frac{\rho'(t)w_3(t)}{\rho(t)}-\frac{w_3^2(t)}{\rho(t)a(t-\sigma_1)}\right]\Bigg]\\
\leq\ & -\frac{\rho(t)R(t)}{z'(t-\sigma_1)4^{\beta-1}}z^\beta(t-\tau_1-\sigma_1)+\left[\frac{\rho'(t)w_1(t)}{\rho(t)}\right.\\
&\left.-\frac{w_1^2(t)}{\rho(t)a(t-\sigma_1)}\right]+b^\beta\left[\frac{\rho'(t)w_2(t)}{\rho(t)}-\frac{w_2^2(t)}{\rho(t)a(t-\sigma_1)}\right]\\
&+\frac{c^\beta}{2^{\beta-1}}\left[\frac{\rho'(t)w_3(t)}{\rho(t)}-\frac{w_3^2(t)}{\rho(t)a(t-\sigma_1)}\right]. \qquad (2.14)
\end{aligned}$$

Since $a(t)$ is nondecreasing and $z''(t)>0$ for $t\geq t_1$, it follows from $(a(t)z''(t))'\leq 0$ that $z'''(t)\leq 0$ for $t\geq t_1$ and therefore by Lemma 2.3, there exists a $k\in(0,1)$ such that

$$\frac{z(t-\sigma_1)}{z'(t-\sigma_1)} \geq \frac{k(t-\sigma_1)}{2}. \tag{2.15}$$

Now $z(t) > 0, z'(t) > 0$ and $z''(t) > 0$ for $t \geq t_1$ imply

$$z(t) = z(t_1) + \int_{t_1}^{t} z'(t)dt \geq (t-t_1)z'(t_1) \geq \frac{dt}{2} \tag{2.16}$$

for some $d > 0$ and for large value of t. From (2.15), (2.16) and $\beta \geq 1$, we have

$$\frac{z^{\beta}(t-\sigma_1)}{z'(t-\sigma_1)} \geq \frac{d^{\beta-1}k(t-\sigma_1)^{\beta}}{2^{\beta}}.$$

Combining the last inequality with (2.14) and then applying the completing the square in the right hand side of the resulting inequality, we obtain

$$\begin{aligned} w_1'(t) + b^{\beta}w_2'(t) + \frac{c^{\beta}}{2^{\beta-1}}w_3'(t) &\leq -\frac{\rho(t)R(t)}{2^{\beta}}\left(\frac{d}{4}\right)^{\beta-1} k(t-\sigma_1)^{\beta} \\ &\quad +(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}})\frac{a(t-\sigma_1)(\rho'(t))^2}{4\rho(t)}. \\ &= -\eta(t)\rho(t) + (1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}})\frac{a(t-\sigma_1)(\rho'(t))^2}{4\rho(t)}. \end{aligned}$$

Integrating the above inequality from $t_2 \geq t_1$ to t, we have

$$\int_{t_2}^{t}\left(\eta(s)\rho(s) - (1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}})\frac{a(s-\sigma_1)(\rho'(s))^2}{4\rho(s)}\right)ds \leq w_1(t_2)+b^{\beta}w_2(t_2)+\frac{c^{\beta}}{2^{\beta-1}}w_3(t_2).$$

Taking lim sup in the last inequality, we get a contradiction to (2.4).

Now, let us assume that Lemma 2.2 (II) holds. Then by Lemma 2.4, we can obtain $\lim_{t\to\infty} x(t) = 0$. This completes the proof. ■

Let $\rho(t) = t$ and $\beta = 1$. Then we can obtain the following corollary to Theorem 2.5.

Corollary 2.6. *Assume that condition* (2.3) *holds,* $\sigma_1 \geq \tau_1$ *and there is a* $t_1 \geq t_0$ *with*

$$\limsup_{t\to\infty} \int_{t_1}^{t}\left(s\eta(s) - \frac{(1+b+c)}{4s}a(s-\sigma_1)\right)ds = \infty \tag{2.17}$$

holds, then every solution $x(t)$ of equation (1.1)*either oscillates or* $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.7. *Assume that condition* (2.3) *holds,* $\sigma_1 \le \tau_1$ *and* $\alpha = \beta \ge 1$. *If there exists a positive real valued function* $\rho(t)$ *and* $t_1 \ge t_0$ *with*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\rho(s)\eta(s) - (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) \frac{a(s-\tau_1)(\rho'(s))^2}{4\rho(s)} \right) ds = \infty \tag{2.18}$$

holds, then every solution $x(t)$ of equation (1.1) *either oscillates or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 2.5, we obtain (2.6). By Lemma 2.2 there are two cases for $z(t)$. Assume Lemma 2.2 (I) holds, for all $t \ge t_1 \ge t_0$. Then we obtain (2.7). By defining

$$w_1(t) = \frac{\rho(t)a(t)z''(t)}{z'(t-\tau_1)},\ w_2(t) = \frac{\rho(t)a(t-\tau_1)z''(t-\tau_1)}{z'(t-\tau_1)},\ w_3(t) = \frac{\rho(t)a(t+\tau_2)z''(t+\tau_2)}{z'(t-\tau_1)}$$

for all $t \ge t_1$, then as in the proof of Theorem 2.5, we obtain

$$\begin{aligned} w_1'(t) + b^\beta w_2'(t) + \frac{c^\beta}{2^{\beta-1}} w_3'(t) \le\ & -\frac{\rho(t)R(t)}{z'(t-\tau_1)4^{\beta-1}} z^\beta(t-\tau_1) + \frac{\rho'(t)w_1(t)}{\rho(t)} \\ & -\frac{w_1^2(t)}{\rho(t)a(t-\tau_1)} + b^\beta \left[\frac{\rho'(t)w_2(t)}{\rho(t)} - \frac{w_2^2(t)}{\rho(t)a(t-\tau_1)} \right] \\ & + \frac{c^\beta}{2^{\beta-1}} \left[\frac{\rho'(t)w_3(t)}{\rho(t)} - \frac{w_3^2(t)}{\rho(t)a(t-\tau_1)} \right]. \end{aligned} \tag{2.19}$$

On the otherhand, by Lemma 2.3, for some $k \in (0,1)$ and for sufficiently large t, we have

$$\frac{z(t-\sigma_1)}{z'(t-\tau_1)} \ge \frac{z(t-\sigma_1)}{z'(t-\sigma_1)} \ge \frac{k(t-\sigma_1)}{2}, \tag{2.20}$$

since $z''(t) \ge 0$ and $\tau_1 \ge \sigma_1$. Combining the inequality (2.20) with (2.19) and then applying the completing the square in the right hand side of the resulting inequality, we have

$$w_1'(t) + b^\beta w_2'(t) + \frac{c^\beta}{2^{\beta-1}} w_3'(t) \le -\eta(t)\rho(t) + (1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}) \frac{a(t-\tau_1)(\rho'(t))^2}{4\rho(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^{t}\left(\eta(s)\rho(s)-(1+b^{\beta}+\frac{c^{\beta}}{2^{\beta-1}})\frac{a(s-\tau_1)(\rho'(s))^2}{4\rho(s)}\right)ds \le w_1(t_2)+b^{\beta}w_2(t_2)+\frac{c^{\beta}}{2^{\beta-1}}w_3(t_2).$$

Taking lim sup on both sides of the above inequality, we obtain a contradiction to (2.18). Assume that Lemma 2.2 (II) holds. Then by Lemma 2.4 we can obtain $\lim_{t\to\infty} x(t)=0$. This completes the proof. ∎

Let $\rho(t)=t$ and $\beta=1$. Then, we can obtain the following corollary to Theorem 2.7.

Corollary 2.8. *Assume that condition* (2.3) *holds,* $\tau_1 \ge \sigma_1$ *and* $\beta=1$. *If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(s\eta(s)-\frac{1+b+c}{4s}a(s-\tau_1)\right)ds=\infty$$

holds, then every solution $x(t)$ *of equation* (1.1) *either oscillates or* $\lim_{t\to\infty} x(t)=0$.

Theorem 2.9. *Assume that* $a(t)\equiv 1, 0<\alpha<1<\beta$ *and* $\sigma_i>\tau_i$ *for* $i=1,2$. *If*

$$\liminf_{t\to\infty}\int_{t}^{t+\sigma-\tau_2}(s-t)^2P^{\eta_1}(s)Q^{\eta_2}(s)ds > 2\eta_1^{\eta_1}\eta_2^{\eta_2}\left(4^{\beta-1}\right)^{\eta_1} \tag{2.21}$$

where $\eta_1=\dfrac{1-\alpha}{\beta-\alpha}, \eta_2=\dfrac{\beta-1}{\beta-\alpha}$ *and* $\sigma=\max(\sigma_1,\sigma_2)$, *then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we assume that there exists a $t_1\ge t_0$ such that $x(t)>0,\ x(t-\sigma_1)>0$ and $x(t-\tau_1)>0$ for all $t\ge t_1$. From equation (1.1), we have

$$z'''(t)=-q(t)x^{\alpha}(t-\sigma_1)-p(t)x^{\beta}(t+\sigma_2)<0 \text{ for all } t\ge t_1.$$

Then as in Lemma 2.2, we have $z''(t)>0$ for all $t\ge t_1$. Define a function $y(t)$ by

$$y(t)=z(t)+b^{\alpha}z(t-\tau_1)+c^{\alpha}z(t+\tau_2). \tag{2.22}$$

Since $z(t) > 0$ and $z''(t) > 0$, we have $y(t) > 0, y''(t) > 0$ and

$$\begin{aligned} y'''(t) &= z'''(t) + b^\alpha z'''(t-\tau_1) + c^\alpha z'''(t+\tau_2) \\ &= -q(t)x^\alpha(t-\sigma_1) - p(t)x^\beta(t+\sigma_2) \\ &\quad + b^\alpha \left[-q(t-\tau_1)x^\alpha(t-\tau_1-\sigma_1) - p(t-\tau_1)x^\beta(t-\tau_1+\sigma_2)\right] \\ &\quad + c^\alpha \left[-q(t+\tau_2)x^\alpha(t+\tau_2-\sigma_1) - p(t+\tau_2)x^\beta(t+\tau_2+\sigma_2)\right] \\ y'''(t) &+ Q(t)\left[x^\alpha(t-\sigma_1) + b^\alpha x^\alpha(t-\tau_1-\sigma_1) + c^\alpha x^\alpha(t+\tau_2-\sigma_1)\right] \\ &\quad + P(t)\left[x^\beta(t+\sigma_2) + b^\alpha x^\beta(t-\tau_1+\sigma_2) + c^\alpha x^\beta(t+\tau_2+\sigma_2)\right] \le 0. \end{aligned}$$

Using Lemma 2.1 and $0 < \alpha < 1 < \beta, b < 1$ and $c < 1$, we get

$$\begin{aligned} y'''(t) + Q(t)\left[x(t-\sigma_1) + b\, x(t-\tau_1-\sigma_1) + c\, x(t+\tau_2-\sigma_1)\right]^\alpha& \\ + P(t)\left[x^\beta(t+\sigma_2) + b^\beta x^\beta(t-\tau_1+\sigma_2) + c^\beta x^\beta(t+\tau_2+\sigma_2)\right] &\le 0. \end{aligned}$$

Now using Lemma 2.1, $c < 1$ and $\beta > 1$, we have

$$\begin{aligned} y'''(t) &+ Q(t)z^\alpha(t-\sigma_1) + P(t)\left[\frac{1}{2^{\beta-1}}(x(t+\sigma_2) + b\, x(t+\sigma_2-\tau_1))^\beta + \frac{c^\beta}{2^{\beta-1}} x^\beta(t+\sigma_2+\tau_2)\right] \le 0 \\ y'''(t) &+ Q(t)z^\alpha(t-\sigma_1) + \frac{P(t)}{4^{\beta-1}}\left[x(t+\sigma_2) + b\, x(t+\sigma_2-\tau_1)) + c\, x(t+\sigma_2+\tau_2)\right]^\beta \le 0 \\ y''' &+ Q(t)z^\alpha(t-\sigma_1) + \frac{P(t)}{4^{\beta-1}} z^\beta(t+\sigma_2) \le 0 \end{aligned}$$

or

$$y'''(t) + Q(t)z^\alpha(t-\sigma) + \frac{P(t)}{4^{\beta-1}} z^\beta(t-\sigma) \le 0. \tag{2.23}$$

Define $u_1 = \eta_1^{-1}\dfrac{P(t)}{4^{\beta-1}} z^\beta(t-\sigma)$ and $u_2 = \eta_2^{-1}Q(t)z^\alpha(t-\sigma)$. Using arithmetic-geometric mean inequality $u_1\eta_1 + u_2\eta_2 \ge u_1^{\eta_1}u_2^{\eta_2}$, we have

$$y'''(t) + \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{P(t)}{4^{\beta-1}}\right)^{\eta_1} Q^{\eta_2}(t)\; z(t-\sigma) \le 0. \tag{2.24}$$

Since $z'(t) > 0$, we see that

$$\begin{aligned} y(t-\sigma) &= z(t-\sigma) + b^{\alpha} z(t-\tau_1-\sigma) + c^{\alpha} z(t+\tau_2-\sigma) \\ &\leq (1+b^{\alpha}+c^{\alpha})\; z(t+\tau_2-\sigma). \end{aligned} \tag{2.25}$$

Using the inequality (2.25) in (2.24), we obtain that $y(t)$ is a positive solution of

$$y'''(t) + \frac{\eta_1^{-\eta_1}\eta_2^{-\eta_2}}{(1+b^{\alpha}+c^{\alpha})}\left(\frac{P(t)}{4^{\beta-1}}\right)^{\eta_1} Q^{\eta_2}(t)\; y(t-\tau_2+\sigma) \leq 0. \tag{2.26}$$

But by [12, Corollary 1], the condition (2.21) implies that equation (2.26) is oscillatory. This contradiction completes the proof. ■

Theorem 2.10. *Assume that* $a(t) \equiv 1, 0 < \beta < 1 < \alpha$ *and* $\sigma_i > \tau_i$ *for* $i = 1, 2$. *If*

$$\liminf_{t\to\infty} \int\limits_{t-(\sigma+\tau_2)/3}^{t} (\sigma+\tau_2)^2 Q^{\eta_1}(s) P^{\eta_2}(s) ds > \frac{1}{e}\left(\frac{3}{2}\right)^2 \dot{\eta}_1^{\eta_1}\eta_2^{\eta_1}\left(4^{\beta-1}\right)^{\eta_1} \tag{2.27}$$

where $\eta_1 = \dfrac{1-\beta}{\alpha-\beta}, \eta_2 = \dfrac{\alpha-1}{\alpha-\beta}$ *and* $\sigma = \max(\sigma_1,\sigma_2)$ *has no increasing solution, then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, let us assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0,\ x(t-\tau_1) > 0$ and $x(t-\sigma_1) > 0$ for all $t \geq t_1$. From equation (1.1), we have $z'''(t) < 0$ for all $t \geq t_1$ and therefore from Lemma 2.2, we have $z''(t) > 0$ for all $t \geq t_1$.

Define a function $y(t)$ by

$$y(t) = z(t) + b^{\alpha} z(t-\tau_1) + c^{\alpha} z(t+\tau_2).$$

Then as in the proof of Theorem 2.9, we have $y(t) > 0,\ y''(t) > 0$ and

$$y'''(t) + \frac{Q(t)}{4^{\alpha-1}} z^{\alpha}(t-\sigma_1) + P(t) z^{\beta}(t+\sigma_2) \leq 0$$

or

$$y'''(t) + \frac{Q(t)}{4^{\alpha-1}} z^{\alpha}(t-\sigma) + P(t) z^{\beta}(t-\sigma) \leq 0. \tag{2.28}$$

Define $u_1 = \eta_1^{-1}\frac{Q(t)}{4^{\alpha-1}}z^{\alpha}(t-\sigma)$ and $u_2 = \eta_2^{-1}P(t)z^{\beta}(t-\sigma)$. Then by using arithmetic-geometric mean inequality $u_1\eta_1 + u_2\eta_2 \geq u_1^{\eta_1}u_2^{\eta_2}$, we have

$$y'''(t) \quad + \quad \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{Q(t)}{4^{\alpha-1}}\right)^{\eta_1} P^{\eta_2}(t)\ z(t-\sigma) \leq 0. \tag{2.29}$$

Since $z'(t) > 0$, we have

$$\begin{aligned} y(t-\sigma) &= z(t-\sigma) + b^{\alpha}z(t-\tau_1-\sigma) + c^{\alpha}z(t+\tau_2-\sigma) \\ &\leq (1 + b^{\alpha} + c^{\alpha})\ z(t+\tau_2-\sigma). \end{aligned} \tag{2.30}$$

Using the inequality (2.30) in (2.29), we obtain that $y(t)$ is a positive solution of

$$y'''(t) + \eta_1^{-\eta_1}\eta_2^{-\eta_2}\left(\frac{Q(t)}{4^{\alpha-1}}\right)^{\eta_1} P^{\eta_2}(t)\ y(t-\tau_2-\sigma) \leq 0. \tag{2.31}$$

But by [12, Corollary 1], the condition (2.27) implies that all solutions of equation (2.31) are oscillatory. This contradiction completes the proof. ■

3. Examples

Example 3.1. Consider the differential equation

$$\left[x(t) + \frac{1}{6e}x(t-1) + \frac{e}{6}x(t+1)\right]''' + e^{2t-6}x^3(t-2) + \frac{2}{3}e^{2t+3}x^3(t+1) = 0,\ t \geq 0. \tag{3.1}$$

Here $a(t) \equiv 1,\ b = \frac{1}{6e},\ c = \frac{e}{6}$ and $b + c = \frac{1+e^2}{6e} < 1,\ \tau_1 = 1,\ \tau_2 = 1,\ \sigma_1 = 2,\ \sigma_2 = 2,$ $\beta = 3,\ q(t) = e^{2t-6},\ p(t) = \frac{2}{3}e^{2t+3},\ \sigma = \max(\sigma_1, \sigma_2).$

$$Q(t) = \min(q(t), q(t-\tau_1), q(t+\tau_2)) = e^{2t-8}$$

$$P(t) = \min(p(t), p(t-\tau_1), p(t+\tau_2)) = \frac{2}{3}e^{2t+1}$$

$$R(t) = Q(t) + P(t) = e^{2t}\left(\frac{1}{e^8} + \frac{2e}{3}\right)$$

$$\eta(s) = \left(\frac{d}{4}\right)^{\beta-1}\frac{k(s-\sigma)^{\beta}}{2^{\beta}},\ R(s) = \left(\frac{d}{4}\right)^2\frac{k}{8}(s-2)^3e^{2s}\left(\frac{1}{e^8} + \frac{2e}{3}\right)$$

By taking $\rho(t) = 1$ one can easily verify that all the conditions of Theorem 2.5 are satisfied. Therefore all the solutions of equation (3.1) are either oscillatory or tend to zero as $t \to \infty$. In particular $x(t) = e^{-t}$ is one solution,since it satisfies equation (3.1), such that $x(t) \to 0$ as $t \to \infty$.

Example 3.2. Consider the differential equation

$$\left[x(t) + \frac{1}{2}x\left(t - \frac{5\pi}{2}\right) + \frac{1}{3}x\left(t + \frac{3\pi}{2}\right)\right]''' + \frac{1}{12}x(t-\pi) + \frac{1}{12}x(t+\pi) = 0,\ t \geq 0. \quad (3.2)$$

Here $a(t) \equiv 1,\ \beta = 1,\ b = 1/3,\ c = 1/3,\ q(t) = \frac{1}{12},\ p(t) = \frac{1}{2},\ R(t) = \frac{1}{6}$

$\sigma_1 = \pi,\ \sigma_2 = \pi,\ \tau_1 = \frac{5\pi}{2},\ \tau_2 = \frac{3\pi}{2},\ \sigma_1 \leq \tau_1,\ \sigma_2 \leq \tau_2,\ \eta(s) = \frac{k(s-\pi)}{12},\ k \in (0,1).$

By taking $\rho(t) = t$, it is easy to see that all the conditions of Corollary 2.8 are satisfied. Therefore all the solutions of equation (3.2) are either oscillatory or tend to zero as $t \to \infty$. In particular $x(t) = \cos t$ is one such solution,since it satisfies equation(3.2), which is an oscillatory solution.

Example 3.3. Consider the differential equation

$$\left[x(t) + \frac{1}{3}x(t-1) + \frac{1}{2}x(t+2)\right]''' + \left(\frac{3}{8t^{3/2}(t-1)^{9/2}} + \frac{1}{8(t-1)^6}\right)x^3(t-1)$$
$$+\frac{3}{16(t+2)^6}x^3(t+2) = 0 \quad (3.3)$$

Here $a(t) = 1,\ \beta = 3 > 1,\ b = \frac{1}{3},\ c = \frac{1}{2},\ q(t) = \frac{3}{8t^{3/2}(t-1)^{9/2}} + \frac{1}{8(t-1)^6},$

$p(t) = \frac{3}{16(t+2)^6},\ R(t) = \frac{3}{8(t+2)^{3/2}(t+1)^{9/2}} + \frac{1}{8(t+2)^6} + \frac{3}{16(t+4)^6}$

$\eta(t) = \left(\frac{d}{4}\right)^2 \frac{k(t-2)^3}{2^3}\left[\frac{3}{8(t+2)^{3/2}(t+1)^{9/2}} + \frac{1}{8(t+2)^6} + \frac{3}{16(t+4)^6}\right]$

By taking $\rho(t) = 1$ one can see that all the conditions of Theorem 2.5 except the conditions 2.3 and 2.4 are satisfied. Therefore all the solutions of equation (3.3) are neither oscillatory nor tend to zero. In particular $x(t) = t^{3/2}$ is one such solution of equation (3.3) such that $\lim_{t\to\infty} x(t) = \infty$.

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