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COEFFICIENT ESTIMATES FOR NEW SUBCLASSES OF ANALYTIC FUNCTIONS WITH RESPECT TO OTHER POINTS

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Abstract. The main purpose of this paper is to derive coefficient estimates for new subclasses of analytic functions with respect to symmetric and conjugate points.

1. Introduction

Let U be the class of functions which are analytic and univalent in the open unit disk $D = \{z : |z| < 1\}$ given by

$$\omega(z) = \sum_{k=1}^{\infty} b_k z^k \tag{1.1}$$

and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| \le 1$, $z \in D$.

Let S denote the class of functions f which are analytic and univalent in D of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in D). \tag{1.2}$$

Let S_s^* be the subclass of S consisting of functions given by (1.2) and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad (z \in D).$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [1].

Also, let S_c^* be the subclass of S consisting of functions given by (1.2) and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z) + \overline{f(\bar{z})}}\right) > 0 \quad (z \in D).$$

These functions are called starlike with respect to conjugate points and were introduced by Ashwah and Thomas [2].

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Motivated by the class S_s^*, Das and Singh [3] discussed the following class C_s, namely convex functions with respect to symmetric points.

Let C_s be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$Re\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0 \quad (z \in D).$$

Suppose that f and g are two analytic functions in U. Then, we say that the function g is subordinate to the function f, and we write

$$g(z) \prec f(z) \quad (z \in D),$$

if there exists a Schwarz function $\varpi(z)$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ such that

$$g(z) = f(\varpi(z)) \quad (z \in D).$$

By applying the above subordination definition, Goel and Mehrok [4] introduced a subclass of S_s^* denoted by $S_s^*(A,B)$.

Let $S_s^*(A,B)$ be the class of functions of the form (1.2) and satisfying the condition

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1;\ z \in D).$$

Also, in the same manner, we give the analogue definitions by extension as follows.

Definition 1.1.

(i) Let $S_c^*(A,B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1,\ z \in D).$$

(ii) Let $C_s(A,B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1,\ z \in D).$$

(iii) Let $C_c(A,B)$ be the subclass of S consisting of functions given by (1.2) satisfying the condition

$$\frac{2(zf'(z))'}{(f(z)+\overline{f(\bar{z})})'} \prec \frac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1,\ z \in D).$$

In this paper, we introduce the class $M_s(\alpha,\mu,A,B)$ consisting of analytic functions f of the form (1.2) and satisfying

$$\frac{2\alpha\mu z^3 f'''(z) + 2(2\alpha\mu+\alpha-\mu)z^2 f''(z) + 2zf'(z)}{\alpha\mu z^2(f(z)-f(-z))'' + (\alpha-\mu)z(f(z)-f(-z))' + (1-\alpha+\mu)(f(z)-f(-z))} \prec \frac{1+Az}{1+Bz}, \tag{1.3}$$

where $-1 \le B < A \le 1,\ 0 \le \mu \le \alpha \le 1$ and $z \in D$.

In addition, we introduce the class $M_c(\alpha,\mu,A,B)$ consisting of analytic functions f of the form (1.2) and satisfying

$$\frac{2\alpha\mu z^3 f'''(z) + 2(2\alpha\mu+\alpha-\mu)z^2 f''(z) + 2zf'(z)}{\alpha\mu z^2(f(z)+\overline{f(\bar z)})'' + (\alpha-\mu)z(f(z)+\overline{f(\bar z)})' + (1-\alpha+\mu)(f(z)+\overline{f(\bar z)})} \prec \frac{1+Az}{1+Bz}, \tag{1.4}$$

where $-1 \le B < A \le 1,\ 0 \le \mu \le \alpha \le 1$ and $z \in D$.

We note that

(i) for $\mu = 0$, $M_s(\alpha,0,A,B) = M_s(\alpha,A,B)$ and $M_c(\alpha,0,A,B) = M_c(\alpha,A,B)$, which were introduced and studied by Selvaraj and Vasanthi [5];

(ii) for $\mu = \alpha = 0$, $M_s(0,0,A,B) = S_s^*(A,B)$ and $M_c(0,0,A,B) = S_c^*(A,B)$;

(iii) for $\mu = 0$ and $\alpha = 1$, $M_s(1,0,A,B) = C_s(A,B)$ and $M_c(1,0,A,B) = C_c(A,B)$.

By the definition of subordination, it follows that $f \in M_s(\alpha,\mu,A,B)$ if and only if

$$\begin{aligned}&\frac{2\alpha\mu z^3 f'''(z) + 2(2\alpha\mu+\alpha-\mu)z^2 f''(z) + 2zf'(z)}{\alpha\mu z^2(f(z)-f(-z))'' + (\alpha-\mu)z(f(z)-f(-z))' + (1-\alpha+\mu)(f(z)-f(-z))}\\ &\quad = \frac{1+A\omega(z)}{1+B\omega(z)} = p(z),\quad \omega(z) \in U\end{aligned} \tag{1.5}$$

and that $f \in M_c(\alpha,\mu,A,B)$ if and only if

$$\begin{aligned}&\frac{2\alpha\mu z^3 f'''(z) + 2(2\alpha\mu+\alpha-\mu)z^2 f''(z) + 2zf'(z)}{\alpha\mu z^2(f(z)+\overline{f(\bar z)})'' + (\alpha-\mu)z(f(z)+\overline{f(\bar z)})' + (1-\alpha+\mu)(f(z)+\overline{f(\bar z)})}\\ &\quad = \frac{1+A\omega(z)}{1+B\omega(z)} = p(z),\quad \omega(z) \in U\end{aligned} \tag{1.6}$$

where

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.7}$$

In the next section, we obtain the coefficient estimates for functions belonging to the classes $M_s(\alpha,\mu,A,B)$ and $M_c(\alpha,\mu,A,B)$.

2. Main results

In order to prove our main results, we shall require the following lemma due to Goel and Mehrok [4].

Lemma 2.1. *If $p(z)$ is given by (1.7), then*

$$|p_n| \le (A-B),\quad n = 1,2,\dots. \tag{2.1}$$

Unless otherwise mentioned, we shall assume in the reminder of this paper that $-1 \le B < A \le 1$, $0 \le \mu \le \alpha \le 1$ and $z \in D$.

Theorem 2.1. *Let* $f \in M_s(\alpha, \mu, A, B)$. *Then, for* $n \ge 1$, *we have*

$$|a_{2n}| \le \frac{(A-B)}{2^n \cdot n![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{n-1}(A-B+2j), \tag{2.2}$$

$$|a_{2n+1}| \le \frac{(A-B)}{2^n \cdot n![1+2n(\alpha-\mu+(2n+1)\alpha\mu)]} \prod_{j=1}^{n-1}(A-B+2j). \tag{2.3}$$

Proof. From (1.5) and (1.7), we have

$$\begin{aligned}
&\big[z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots\big]\\
&\quad+(2\alpha\mu+\alpha-\mu)[2a_2z^2+6a_3z^3+12a_4z^4+20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots]\\
&\quad+\alpha\mu[6a_3z^3+24a_4z^4+60a_5z^5+\cdots+(2n-1)2n(2n+1)a_{2n+1}z^{2n+1}+\cdots]\\
&=\Big[(1+\alpha-\mu)[z+a_3z^3+a_5z^5+\cdots+a_{2n-1}z^{2n-1}+a_{2n+1}z^{2n+1}+\cdots]\\
&\quad+(\alpha-\mu)[z+3a_3z^3+5a_5z^5+\cdots+(2n-1)a_{2n-1}z^{2n-1}+(2n+1)a_{2n+1}z^{2n+1}+\cdots]\\
&\quad+\alpha\mu[6a_3z^3+20a_5z^5+\cdots+2n(2n+1)a_{2n+1}z^{2n+1}+\cdots]\Big]\\
&\quad\cdot[1+p_1z+p_2z^2+p_3z^3+p_4z^4+p_5z^5+\cdots+p_{2n-1}z^{2n-1}+p_{2n}z^{2n}+\cdots].
\end{aligned}$$

Equating the coefficients of like powers of z, we obtain

$$2[1+(\alpha-\mu+2\alpha\mu)]a_2 = p_1,\quad 2[1+2(\alpha-\mu+3\alpha\mu)]a_3 = p_2 \tag{2.4}$$

$$4[1+3(\alpha-\mu+4\alpha\mu)]a_4 = p_3+[1+2(\alpha-\mu+3\alpha\mu)]a_3p_1 \tag{2.5}$$

$$4[1+4(\alpha-\mu+5\alpha\mu)]a_5 = p_4+[1+2(\alpha-\mu+3\alpha\mu)]a_3p_2 \tag{2.6}$$

$$\begin{aligned}2n[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]a_{2n} &= p_{2n-1}+[1+2(\alpha-\mu+3\alpha\mu)]a_3p_{2n-3}+\cdots\\ &\quad+[1+(2n-2)(\alpha-\mu+(2n-1)\alpha\mu)]a_{2n-1}p_1\end{aligned} \tag{2.7}$$

$$\begin{aligned}(2n+1)[1+2n(\alpha-\mu+(2n+1)\alpha\mu)]a_{2n+1} &= p_{2n}+[1+2(\alpha-\mu+3\alpha\mu)]a_3p_{2n-2}+\cdots\\ &\quad+[1+(2n-2)(\alpha-\mu+(2n-1)\alpha\mu)]a_{2n-1}p_2\end{aligned} \tag{2.8}$$

By using Lemma 2.1 and (2.4), we get

$$|a_2| \le \frac{A-B}{2[1+(\alpha-\mu+2\alpha\mu)]},\quad |a_3| \le \frac{A-B}{2[1+2(\alpha-\mu+3\alpha\mu)]}. \tag{2.9}$$

Again, making use of (2.1), in conjunction with (2.9), we find from (2.5) and (2.6) that

$$|a_4| \le \frac{(A-B)(A-B+2)}{2\cdot 4\cdot[1+3(\alpha-\mu+4\alpha\mu)]},$$

$$|a_5| \le \frac{(A-B)(A-B+2)}{2\cdot 4\cdot[1+4(\alpha-\mu+5\alpha\mu)]}.$$

It follows that (2.2) and (2.3) hold for $n=1,2$. Next, we prove (2.2) by induction.

Equation (2.7) together with Lemma 2.1 yields

$$|a_{2n}| \le \frac{(A-B)}{2n[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\left\{1+\sum_{k=1}^{n-1}[1+2k(\alpha-\mu+(2k+1)\alpha\mu)]|a_{2k+1}|\right\}. \tag{2.10}$$

We suppose that (2.2) holds for $k=3,4,\dots,(n-1)$.

Then from (2.10), we have

$$|a_{2n}| \le \frac{(A-B)}{2n[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right]. \tag{2.11}$$

In order to complete the proof, it is sufficient to show that

$$\begin{aligned}&\frac{(A-B)}{2m[1+(2m-1)(\alpha-\mu+2m\alpha\mu)]}\left[1+\sum_{k=1}^{m-1}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\\ &\quad=\frac{(A-B)}{2^m\cdot m![1+(2m-1)(\alpha-\mu+2m\alpha\mu)]}\prod_{j=1}^{m-1}(A-B+2j)\quad (m=3,4,\dots,n),\end{aligned} \tag{2.12}$$

which is valid for $m=3$.

Let us assume that (2.12) is true for all m, $3<m\le(n-1)$. Then from (2.11), we get

$$\begin{aligned}&\frac{(A-B)}{2n[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right]\\
&=\left(\frac{n-1}{n}\right)\left(\frac{(A-B)}{2(n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\left(1+\sum_{k=1}^{n-2}\frac{(A-B)}{2^k k!}\prod_{j=1}^{k-1}(A-B+2j)\right)\right)\\
&\quad+\frac{(A-B)}{2n[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\cdot\frac{(A-B)}{2^{n-1}\cdot(n-1)!}\prod_{j=1}^{n-2}(A-B+2j)\\
&=\left(\frac{n-1}{n}\right)\frac{(A-B)}{2^{n-1}\cdot(n-1)![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\prod_{j=1}^{n-2}(A-B+2j)\\
&\quad+\frac{(A-B)}{2n[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\cdot\frac{(A-B)}{2^{n-1}\cdot(n-1)!}\prod_{j=1}^{n-2}(A-B+2j)\\
&=\frac{(A-B)}{2n(n-1)!2^{n-1}[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\prod_{j=1}^{n-2}(A-B+2j)(A-B+2(n-1))\\
&=\frac{(A-B)}{2^n\cdot n![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\prod_{j=1}^{n-1}(A-B+2j).\end{aligned}$$

Thus (2.12) holds for $m=n$ and hence (2.2) follows. Similarly, we can prove (2.3).

Theorem 2.2. *Let $f \in M_c(\alpha,\mu,A,B)$. Then, for $n \geq 1$, we have*

$$|a_{2n}| \leq \frac{(A-B)}{(2n-1)![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{2n-2}(A-B+j), \tag{2.13}$$

$$|a_{2n+1}| \leq \frac{(A-B)}{(2n)![1+2n(\alpha-\mu+(2n+1)\alpha\mu)]} \prod_{j=1}^{2n-1}(A-B+j). \tag{2.14}$$

Proof. From (1.6) and (1.7), we have

$$\begin{aligned}
&[z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots]\\
&\quad +(2\alpha\mu+\alpha-\mu)[2a_2z^2+6a_3z^3+12a_4z^4+20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots]\\
&\quad +\alpha\mu[6a_3z^3+24a_4z^4+60a_5z^5+\cdots+(2n-1)2n(2n+1)a_{2n+1}z^{2n+1}+\cdots]\\
&= \Big[(1+\alpha-\mu)[z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots+a_{2n}z^{2n}+\cdots]\\
&\quad +(\alpha-\mu)[z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+\cdots+2na_{2n}z^{2n}+\cdots]\\
&\quad +\alpha\mu[2a_2z^2+6a_3z^3+12a_4z^4+20a_5z^5+\cdots+(2n-1)2na_{2n}z^{2n}+\cdots]\Big]\\
&\quad \cdot[1+p_1z+p_2z^2+p_3z^3+p_4z^4+p_5z^5+\cdots+p_{2n-1}z^{2n-1}+\cdots]
\end{aligned}$$

Equating the coefficients of like powers of z, we obtain

$$[1+(\alpha-\mu+2\alpha\mu)]a_2 = p_1, \quad 2[1+2(\alpha-\mu+3\alpha\mu)]a_3 = p_2+[1+(\alpha-\mu+2\alpha\mu)]a_2p_1 \tag{2.15}$$

$$3[1+3(\alpha-\mu+4\alpha\mu)]a_4 = p_3+[1+(\alpha-\mu+2\alpha\mu)]a_2p_2+[1+2(\alpha-\mu+3\alpha\mu)]a_3p_1 \tag{2.16}$$

$$\begin{aligned}4[1+4(\alpha-\mu+5\alpha\mu)]a_5 = {}& p_4+[1+(\alpha-\mu+2\alpha\mu)]a_2p_3+[1+2(\alpha-\mu+3\alpha\mu)]a_3p_2\\ &+[1+3(\alpha-\mu+4\alpha\mu)]a_4p_1\end{aligned} \tag{2.17}$$

$$\begin{aligned}(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]a_{2n} = {}& p_{2n-1}+[1+(\alpha-\mu+2\alpha\mu)]a_2p_{2n-2}+\cdots\\ &+[1+(2n-2)(\alpha-\mu+(2n-1)\alpha\mu)]a_{2n-1}p_1\end{aligned} \tag{2.18}$$

$$\begin{aligned}(2n)[1+2n(\alpha-\mu+(2n+1)\alpha\mu)]a_{2n+1} = {}& p_{2n}+[1+(\alpha-\mu+2\alpha\mu)]a_2p_{2n-1}+\cdots\\ &+[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]a_{2n}p_1\end{aligned} \tag{2.19}$$

By using Lemma 2.1 and (2.15), we obtain

$$|a_2| \leq \frac{A-B}{[1+(\alpha-\mu+2\alpha\mu)]}, \quad |a_3| \leq \frac{(A-B)(A-B+1)}{2[1+2(\alpha-\mu+3\alpha\mu)]}. \tag{2.20}$$

Again, making use of (2.1), in conjunction with (2.20), we find from (2.16) and (2.17) that

$$\begin{aligned}
|a_4| &\leq \frac{(A-B)(A-B+1)(A-B+2)}{2\cdot 3\cdot[1+3(\alpha-\mu+4\alpha\mu)]},\\
|a_5| &\leq \frac{(A-B)(A-B+1)(A-B+2)(A-B+3)}{2\cdot 3\cdot 4\cdot[1+4(\alpha-\mu+5\alpha\mu)]}.
\end{aligned}$$

It follows that (2.13) and (2.14) hold for $n = 1,2$. Next, we prove (2.13) by induction.

Equation (2.18) together with Lemma 2.1 yields

$$|a_{2n}| \le \frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\Big\{1+\sum_{k=1}^{n-1}[1+(2k-1)(\alpha-\mu+2k\alpha\mu)]|a_{2k}| + \sum_{k=1}^{n-1}[1+2k(\alpha-\mu+(2k+1)\alpha\mu)]|a_{2k+1}|\Big\}. \tag{2.21}$$

We suppose that (2.13) holds for $k = 3,4,\dots,(n-1)$.

Then from (2.21), we obtain

$$|a_{2n}| \le \frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\Big[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j) + \sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\Big]. \tag{2.22}$$

In order to complete the proof, it is sufficient to show that

$$\begin{aligned}&\frac{(A-B)}{(2m-1)[1+(2m-1)(\alpha-\mu+2m\alpha\mu)]}\\ &\quad\cdot\left[1+\sum_{k=1}^{m-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{m-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\\ &= \frac{(A-B)}{(2m-1)![1+(2m-1)(\alpha-\mu+2m\alpha\mu)]}\prod_{j=1}^{2m-2}(A-B+j)\quad (m=3,4,\dots,n),\end{aligned} \tag{2.23}$$

which is valid for $m = 3$.

Let us assume that (2.23) is true for all m, $3 < m \le (n-1)$. Then from (2.22), we get

$$\begin{aligned}&\frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\\ &\quad\cdot\left[1+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-1}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\right]\\ &= \left(\frac{2n-3}{2n-1}\right)\Big[\frac{(A-B)}{(2(n-1)-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\\ &\quad\cdot\Big(1+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k-1)!}\prod_{j=1}^{2k-2}(A-B+j)+\sum_{k=1}^{n-2}\frac{(A-B)}{(2k)!}\prod_{j=1}^{2k-1}(A-B+j)\Big)\Big]\\ &\quad+\frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\cdot\frac{(A-B)}{(2(n-1)-1)!}\prod_{j=1}^{2n-4}(A-B+j)\\ &\quad+\frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]}\cdot\frac{(A-B)}{(2(n-1))!}\prod_{j=1}^{2n-3}(A-B+j)\end{aligned}$$

$$= \left(\frac{2n-3}{2n-1}\right) \frac{(A-B)}{(2(n-1)-1)![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{2n-4} (A-B+j)$$
$$+ \frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \cdot \frac{(A-B)}{(2(n-1)-1)!} \prod_{j=1}^{2n-4} (A-B+j)$$
$$+ \frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \cdot \frac{(A-B)}{(2(n-1))!} \prod_{j=1}^{2n-3} (A-B+j)$$
$$= \frac{(A-B)}{(2n-1)(2(n-1)-1)![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{2n-4} (A-B+j)(A-B+2n-3)$$
$$+ \frac{(A-B)}{(2n-1)[1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \cdot \frac{(A-B)}{(2(n-1))!} \prod_{j=1}^{2n-3} (A-B+j)$$
$$= \frac{(A-B)}{(2n-1)![1+(2n-1)(\alpha-\mu+2n\alpha\mu)]} \prod_{j=1}^{2n-2} (A-B+j).$$

Thus (2.23) holds for $m = n$ and hence (2.13) follows. Similarly, we can prove (2.14).

Remark 2.1. Taking $\mu = 0$ in Theorems 2.1 and 2.2, we obtain the results obtained by Selvaraj and Vasanthi [5, Theorems 3.1 and 3.2, respectively].

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