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COMPREHENSIVE FAMILY OF UNIFORMLY ANALYTIC FUNCTIONS

B. A. FRASIN

Abstract. We introduce the subclass $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta)$ of analytic functions with negative coefficients. Coefficient inequalities, distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta)$ are obtained. We also determine integral operators for functions in this class and some properties involving modified Hadamard products of several functions belonging to the class $\mathcal{U}^*_{\mathcal{T}}(\Phi, \Psi, \alpha, \beta)$.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in Δ. We define the family $\mathcal{U}(\Phi, \Psi; \alpha, k)$ consisting of the functions $f \in \mathcal{A}$ so that

$$\mathrm{Re}\left\{\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - \alpha\right\} > k\left|\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)} - 1\right|, \qquad (z \in \Delta) \tag{1.2}$$

where $-1 \le \alpha < 1$, $k \ge 0$, $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic in Δ with the conditions $\lambda_n \ge 0$, $\mu_n \ge 0$, $\lambda_n \ge \mu_n$ *for* $n \ge 2$, and $f(z) * \Psi(z) \ne 0$.

The operator "$*$" stands for the Hadamard product or convolution of two power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ given by $f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n$.

The family $\mathcal{U}(\Phi, \Psi; \alpha, k)$ is of special interest because it reduces to various classes of well-known functions as well as many new ones. For example

$$\mathcal{U}\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; 0, k\right) = k\text{–}\mathcal{ST} \equiv \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|$$

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and

$$\mathcal{U}\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; 0, k\right) = k\text{–}\mathcal{UCV} \equiv \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|.$$

are, respectively, the subclasses of $\mathcal{S}$ consisting of functions which are *k- starlike* and *k-uniformly convex* in Δ introduced by Kanas and Winsiowska [1, 2](see also the work of Kanas and Srivastava [3] and Gangadharan et al.[4]). In particular, when $k = 1$, we obtain $1\text{–}\mathcal{ST} \equiv \mathcal{SP}$ and $1\text{–}\mathcal{UCV} \equiv \mathcal{UCV}$, where $\mathcal{SP}$ and $\mathcal{UCV}$ are the familiar classes of uniformly convex functions and parabolic starlike functions in Δ, respectively (see, for details, [5-8]). Another subclass is the subclass

$$\mathcal{U}\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, 1\right) = \mathcal{UCV}(\alpha) \equiv \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) - \alpha > k\left|\frac{zf''(z)}{f'(z)}\right|.$$

of uniformly convex functions of order α which is defined by Rønning [9].

Let $\mathcal{T}$ denotes the subclass of $\mathcal{S}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0) \tag{1.3}$$

Further, let

$$\mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, \beta) = \mathcal{U}(\Phi, \Psi, \alpha, \beta) \cap \mathcal{T} \tag{1.4}$$

In the present paper, we prove various coefficient inequalities, distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, k)$. We also determine some properties involving modified Hadamard products and integral operator of several functions in this class.

2. Coefficient Inequalities

Theorem 1. *Given $\alpha(1 \leq \alpha < 1)$, $k \geq 0$. If*

$$\sum_{n=2}^{\infty} [(1+k)\lambda_n - (\alpha+k)\mu_n] |a_n| \leq 1-\alpha \tag{2.1}$$

then $f(z) \in \mathcal{U}(\Phi, \Psi; \alpha, k)$.

Proof. It suffices to show that

$$k\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right| - \operatorname{Re}\left\{\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right\} \leq 1-\alpha$$

We have

$$k\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right| - \operatorname{Re}\left\{\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right\} \leq (1+k)\left|\frac{f(z)*\Phi(z)}{f(z)*\Psi(z)} - 1\right|$$

$$\leq \frac{(1+k)\sum\limits_{n=2}^{\infty}(\lambda_n - \mu_n)|a_n||z|^n}{1-\sum\limits_{n=2}^{\infty}\mu_n|a_n||z|^n} \leq \frac{(1+k)\sum\limits_{n=2}^{\infty}(\lambda_n - \mu_n)|a_n|}{1-\sum\limits_{n=2}^{\infty}\mu_n|a_n|}.$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}[(1+k)\lambda_n-(\alpha+k)\mu_n]\,|a_n|\le(1-\alpha)$$

which is equivalent to (2.1).

Now we prove that the condition (2.1) is also necessary for $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$.

Theorem 2. *$f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi,\alpha,k)$ for $\alpha(-1\le\alpha<1)$ and $k\ge0$, iff*

$$\sum_{n=2}^{\infty}[(1+k)\lambda_n-(\alpha+k)\mu_n]a_n\le1-\alpha. \tag{2.2}$$

The result (2.2) is sharp.

Proof. In view of Theorem 1, we need only to prove the necessity. Let $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty}\lambda_n a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\mu_n a_n z^{n-1}}-\alpha\ge\frac{1-\sum\limits_{n=2}^{\infty}(\lambda_n-\mu_n)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\mu_n a_n z^{n-1}}.$$

Letting $z\to1$ along the real axis, we obtain the inequality (2.2).

The result (2.2) is sharp for the function

$$f(z)=z-\frac{1-\alpha}{(1+\beta)\lambda_n-(\alpha+\beta)\mu_n}z^n\qquad(n\ge2). \tag{2.3}$$

For the notational convenience we shall henceforth denote

$$\sigma_n(\alpha,k)=(1+k)\lambda_n-(\alpha+k)\mu_n\qquad(n\ge2). \tag{2.4}$$

3. Growth and Distortion Theorems

Theorem 3. *Let the function $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. If $\{\sigma_n(\alpha,k)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then, for $|z|=r<1$*

$$r-\frac{1-\alpha}{\sigma_2(\alpha,k)}r^2\le|f(z)|\le r+\frac{1-\alpha}{\sigma_2(\alpha,k)}r^2 \tag{3.1}$$

and if $\{\sigma_n(\alpha,k)/n\}_{n=2}^{\infty}$is a non-decreasing sequence, then, for $|z|=r<1$

$$1-\frac{2(1-\alpha)}{\sigma_2(\alpha,\beta)}r\le|f'(z)|\le1+\frac{2(1-\alpha)}{\sigma_2(\alpha,\beta)}r. \tag{3.2}$$

The results (3.1) *and* (3.2) *are sharp for the function* $f(z)$ *given*

$$f(z) = z - \frac{1-\alpha}{\sigma_2(\alpha,k)} z^2 \qquad (z = \pm r). \tag{3.3}$$

Proof. In view of Theorem 2, we note that $\sigma_2(\alpha,k) \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \sigma_n(\alpha,k) a_n \leq 1-\alpha$. Thus

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1-\alpha}{\sigma_2(\alpha,k)} r^2 \tag{3.4}$$

Similarly,

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1-\alpha}{\sigma_2(\alpha,k)} r^2. \tag{3.5}$$

Also from Theorem 2, we have $\frac{\sigma_2(\alpha,\beta)}{2} \sum_{n=2}^{\infty} n a_n \leq \sum_{n=2}^{\infty} n \sigma_n(\alpha,\beta) a_n \leq 1-\alpha$. Thus

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n a_n \geq 1 - \frac{2(1-\alpha)}{\sigma_2(\alpha,k)} r.$$

On the other hand,

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n a_n \leq 1 + \frac{2(1-\alpha)}{\sigma_2(\alpha,k)} r.$$

This completes the proof.

Corollary 1. *The disk* $|z| < 1$ *is mapped onto a domain that contains the disk* $|w| < \frac{\sigma_2(\alpha,k)-(1-\alpha)}{\sigma_2(\alpha,k)}$ *by any* $f(z) \in \mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. *The theorem is sharp with extermal function* $f(z)$ *given by* (3.3).

Proof. The proof follow upon letting $r \to 1$ in (3.1).

4. Closure Theorems

In this section, we shall prove that the class $\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$ is closed under convex linear combinations.

Theorem 4. *Let the function* $f_i(z)$, $i = 1, 2, \ldots, m$, *defined by*

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \geq 0) \tag{4.1}$$

for $z \in \Delta$, *be in the class* $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, k)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{n=2}^{\infty} \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) z^n \tag{4.2}$$

also belongs to the class $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi, \alpha, k)$.

Proof. Let $f_i(z) \in \mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, k)$,it follows from Theorem 2 that

$$\sum_{n=2}^{\infty} \sigma_n(\alpha, k) a_{n,i} \leq 1 - \alpha \qquad (\, i = 1, 2, \ldots, m). \tag{4.3}$$

Therefore,

$$\sum_{n=2}^{\infty} \sigma_n(\alpha, k) \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) \tag{4.4}$$

$$= \frac{1}{m} \sum_{i=1}^{m} \left(\sum_{n=2}^{\infty} \sigma_n(\alpha, k) a_{n,i} \right) \leq 1 - \alpha. \tag{4.5}$$

Hence by Theorem 2, $h(z) \in \mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, k)$.

With the aid of Theorem 2, we can prove the following

Theorem 5. *Let the functions* $f_i(z)$ *be defined by* (4.1) *be in the class* $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi, \alpha, k)$ *for every* $i = 1, 2, \ldots, m$. *Then the functions*

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geq 0) \tag{4.6}$$

is also in the same class $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi, \alpha, k)$ *where* $\sum\limits_{i=1}^{m} c_i = 1$.

Theorem 6. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1 - \alpha}{\sigma_n(\alpha, k)} z^n \qquad (n \geq 2). \tag{4.7}$$

Then $f(z) \in \mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \alpha, k)$ *if and only if it can be expressed in the form:*

$$f(z) = \sum_{n=1}^{\infty} \delta_n f_n(z) \tag{4.8}$$

where

$$\delta_n \geq 0 \quad (n \geq 1) \textit{and} \ \sum_{n=1}^{\infty} \delta_n = 1.$$

Proof. Assume that

$$f(z)=\sum_{n=1}^{\infty}\delta_n f_n(z)=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{\sigma_n(\alpha,k)}\delta_n z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty}\frac{\sigma_n(\alpha,k)}{1-\alpha}.\frac{1-\alpha}{\sigma_n(\alpha,k)}\delta_n=\sum_{n=2}^{\infty}\delta_n=1-\delta_1\leq 1.$$

So by Theorem 1, $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$.

Conversely, assume that the function $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. Then

$$a_n\leq\frac{1-\alpha}{\sigma_n(\alpha,k)}\qquad(n\geq 2).$$

Setting

$$\delta_n=\frac{\sigma_n(\alpha,k)}{1-\alpha}a_n\qquad(n\geq 2),$$

and

$$\delta_1=1-\sum_{n=2}^{\infty}\delta_n$$

we can see that $f(z)$ can be expressed in the form (4.8). This completes the proof of the Theorem 6.

5. Radii of Close-to-convexity, Starlikeness and Convexity

Theorem 7. *Let the function $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. Then $f(z)$ is close-to-convex of order $\rho(0\leq\rho<1)$ in $|z|<r_1$, where*

$$r_1=\inf_n\left[\frac{\sigma_n(\alpha,k)(1-\rho)}{n(1-\alpha)}\right]^{1/(n-1)}\qquad(n\geq 2).\tag{5.1}$$

The result is sharp, the extermal function $f(z)$ being given by (2.3).

Proof. Let $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$, then the function $f(z)$ defined by (1.3) is close-to-convex of order ρ in $|z|<r_1$, provided that

$$\begin{aligned}|f'(z)-1|&=\left|\sum_{n=2}^{\infty}na_nz^{n-1}\right|\\&\leq\sum_{n=2}^{\infty}na_n|z|^{n-1}\leq 1-\rho,\qquad(|z|<r_1;\ 0\leq\rho<1),\end{aligned}\tag{5.2}$$

where r_1 is given by (5.1). But, by Theorem 2, (5.2) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1} \leq \frac{\sigma_n(\alpha,k)}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{\sigma_n(\alpha,k)(1-\rho)}{n(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.3}$$

Theorem 7 follows easily from (5.3).

Theorem 8. *Let the function $f(z) \in \mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. Then $f(z)$ is starlike of order $\rho(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_n \left[\frac{\sigma_n(\alpha,k)(1-\rho)}{(n-\rho)(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.4}$$

The result is sharp, with the extermal function $f(z)$ given by (2.3).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho$ for $|z| < r_2$, where r_2 is given by (5.4). From (1.3) we find that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho$ if

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1} \leq 1 \tag{5.5}$$

But, by Theorem 2, (5.5) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z|^{n-1} \leq \frac{\sigma_n(\alpha,k)}{1-\alpha}$$

that is, if

$$|z| \leq \left[\frac{\sigma_n(\alpha,k)(1-\rho)}{(n-\rho)(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.6}$$

Theorem 8 follows easily from (5.6).

Corollary 2. *Let the function $f(z) \in \mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. Then $f(z)$ is convex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = \inf_n \left[\frac{\sigma_n(\alpha,k)(1-\rho)}{n(n-\rho)(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.7}$$

The result is sharp, with the extermal function $f(z)$ given by (2.3).

6. Integral Operators

Theorem 9. *Let the functions $f(z)$ defined by* (1.3) *be in the class $\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$, and let c be real number such that $c>-1$. Then the function $F(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$, where*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{6.1}$$

Proof. From (6.1), it follows that

$$F(z)=z-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)a_nz^n.$$

Therefore,

$$\sum_{n=2}^{\infty}\left(\frac{\sigma_n(\alpha,k)}{1-\alpha}\right)\left(\frac{c}{n+c+1}\right)a_n\le\sum_{n=2}^{\infty}\left(\frac{\sigma_n(\alpha,k)}{1-\alpha}\right)a_n\le 1$$

since $f(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. Hence, by Theorem 2, $F(z)\in\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$

Theorem 10. *Let the function $F(z)=z-\sum_{n=2}^{\infty}a_nz^n$ $(a_n\ge 0)$ be in the class $\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$ and let c be a real number such that $c>-1$. Then the function given by* (6.1) *is univalent in $|z|<r_4$, where*

$$r_4=\inf_n\left[\frac{\sigma_n(\alpha,k)(c+1)}{n(c+n)\,|\beta|}\right]^{1/(n-1)}\qquad(n\ge 2). \tag{6.2}$$

The result is sharp.

Proof. From (6.1), we have

$$f(z)=\frac{z^{1-c}(z^cF(z))'}{c+1}=z-\sum_{n=2}^{\infty}\left(\frac{c+n}{c+1}\right)a_nz^n.$$

In order to obtain the required result, it suffices to show that $|f'(z)-1|<1$ whenever $|z|<r_4$, where r_4 is given by (6.2). Now

$$|f'(z)-1|\le\sum_{n=2}^{\infty}n\left(\frac{c+n}{c+1}\right)a_n\,|z|^{n-1}.$$

Thus $|f'(z)-1|<1$ if

$$\sum_{n=2}^{\infty}n\left(\frac{c+n}{c+1}\right)a_n\,|z|^{n-1}<1. \tag{6.3}$$

But from Theorem 2, (6.3) will be satisfied if

$$\frac{n(c+n)}{c+1}|z|^{n-1} < \frac{\sigma_n(\alpha,k)}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{\sigma_n(\alpha,k)(c+1)}{n(c+n)(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2).$$

Therefore, $f(z)$ is univalent in $|z| < r_4$.

The result is sharp for the function

$$f(z) = z - \frac{(c+n)(1-\alpha)}{\sigma_n(\alpha,k)(c+1)}z^n \qquad (n \geq 2). \tag{6.4}$$

7. Modified Hadamard Product

Let the functions $f_i(z)$ $(i = 1,2)$ be defined by (4.1), then we define the modified Hadamard product of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. \tag{7.1}$$

Employing the technique used earlier by Schild and Silverman [10], we prove the following

Theorem 11. *Let each of the functions $f_j(z)$ $(j = 1,2)$ defined by* (4.1) *be in the class* $\mathcal{U}_{\mathcal{T}}(\Phi,\Psi;\alpha,k)$. Let

$$\Delta(n) = \frac{(\sigma_n(\alpha,k))^2 - (1-\alpha)^2[(1+k)\lambda_n - k\mu_n]}{(\sigma_n(\alpha,k))^2 - (1-\alpha)^2\mu_n}, \tag{7.2}$$

If $\Delta(n)$ *is an increasing function of* $n (n \geq 2)$, then $(f_1 * f_2)(z) \in \mathcal{U}_{\mathcal{T}}(\Phi,\Psi,\gamma,k)$, *for*

$$\gamma = \frac{(\sigma_2(\alpha,k))^2 - (1-\alpha)^2[(1+k)\lambda_2 - k\mu_2]}{(\sigma_2(\alpha,k))^2 - (1-\alpha)^2\mu_2}. \tag{7.3}$$

The result is sharp.

Proof. We need to find the largest γ such that

$$\sum_{n=2}^{\infty} \frac{\sigma_n(\gamma,k)}{1-\gamma}a_{n,1}a_{n,2} \leq 1.$$

From Theorem 2, we have $\sum\limits_{n=2}^{\infty} \frac{\sigma_n(\alpha,k)}{1-\alpha} a_{n,1} \leq 1$ and $\sum\limits_{n=2}^{\infty} \frac{\sigma_n(\alpha,k)}{1-\alpha} a_{n,2} \leq 1$, by the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{\sigma_n(\alpha,k)}{1-\alpha} \sqrt{a_{n,1}a_{n,2}} \leq 1.$$

Thus it is sufficient to show that

$$\frac{\sigma_n(\gamma,k)}{1-\gamma} a_{n,1}a_{n,2} \leq \frac{\sigma_n(\alpha,k)}{1-\alpha} \sqrt{a_{n,1}a_{n,2}}, \qquad (n \geq 2)$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{\sigma_n(\alpha,k)(1-\gamma)}{\sigma_n(\gamma,k)(1-\alpha)} \qquad (n \geq 2).$$

Note that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{1-\alpha}{\sigma_n(\alpha,k)} \qquad (n \geq 2).$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{\sigma_n(\alpha,k)} \leq \frac{\sigma_n(\alpha,k)(1-\gamma)}{\sigma_n(\gamma,k)(1-\alpha)} \qquad (n \geq 2)$$

or, equivalently

$$\gamma \leq \frac{(\sigma_n(\alpha,k))^2 - (1-\alpha)^2[(1+k)\lambda_n - k\mu_n]}{(\sigma_n(\alpha,k))^2 - (1-\alpha)^2\mu_n} = \Delta(n) \tag{7.4}$$

Since $\Delta(n)$ is an increasing function of $n (n \geq 2)$, letting $n = 2$ in (7.4), we obtain

$$\gamma \leq \Delta(2) = \frac{(\sigma_2(\alpha,k))^2 - (1-\alpha)^2[(1+k)\lambda_2 - k\mu_2]}{(\sigma_2(\alpha,k))^2 - (1-\alpha)^2\mu_2},$$

which proves the main assertion of Theorem 12.

Finally, by taking the functions

$$f_j(z) = z - \frac{1-\alpha}{\sigma_2(\alpha,k)} z^2 \qquad (j = 1, 2) \tag{7.5}$$

we can see the result is sharp.

Theorem 12. *Let each of the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $\mathcal{U_T}(\Phi, \Psi; \alpha, k)$. Let*

$$\Omega(n) = \frac{\frac{1}{2}(\sigma_n(\alpha,k))^2 - [(1+k)\lambda_n - k\mu_n](1-\alpha)^2}{\frac{1}{2}(\sigma_n(\alpha,k))^2 - \mu_n(1-\alpha)^2}, \tag{7.6}$$

If $\Omega(n)$ *is an increasing function of* $n (n \geq 2)$*, then the function*

$$h(z) = z - \sum_{n=2}^{\infty} (a_{n,1}^2 + a_{n,2}^2) z^n \tag{7.7}$$

belongs to the class $\mathcal{U}_{\mathcal{T}}(\Phi, \Psi; \tau, k)$*, where*

$$\tau = \frac{\frac{1}{2}(\sigma_2(\alpha,k))^2 - [(1+k)\lambda_2 - k\mu_2](1-\alpha)^2}{\frac{1}{2}(\sigma_2(\alpha,k))^2 - \mu_2(1-\alpha)^2}. \tag{7.8}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ *defined by* (7.5).

Proof. From Theorem 1, we have

$$\sum_{n=2}^{\infty} \left[\frac{\sigma_n(\alpha,k)}{1-\alpha}\right]^2 a_{n,1}^2 \leq \sum_{n=2}^{\infty} \left[\frac{\sigma_n(\alpha,k)}{1-\alpha} a_{n,1}\right]^2 \leq 1 \tag{7.9}$$

and

$$\sum_{n=2}^{\infty} \left[\frac{\sigma_n(\alpha,\beta)}{1-\alpha}\right]^2 a_{n,2}^2 \leq \sum_{k=2}^{\infty} \left[\frac{\sigma_n(\alpha,\beta)}{1-\alpha} a_{n,2}\right]^2 \leq 1. \tag{7.10}$$

It follows from (7.9) and (7.10) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left[\frac{\sigma_n(\alpha,k)}{1-\alpha}\right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1$$

Therefore, we need to find the largest τ such that

$$\frac{\sigma_n(\tau,k)}{1-\tau} \leq \frac{1}{2} \left[\frac{\sigma_n(\alpha,k)}{1-\alpha}\right]^2 \qquad (n \geq 2)$$

that is,

$$\tau \leq \frac{\frac{1}{2}(\sigma_n(\alpha,k))^2 - [(1+k)\lambda_n - k\mu_n](1-\alpha)^2}{\frac{1}{2}(\sigma_n(\alpha,k))^2 - \mu_n(1-\alpha)^2} = \Omega(n).$$

Since $\Omega(n)$ is an increasing function of $n(n \geq 2)$,we readily have

$$\tau \leq \Omega(2) = \frac{\frac{1}{2}(\sigma_2(\alpha,k))^2 - [(1+k)\lambda_2 - k\mu_2](1-\alpha)^2}{\frac{1}{2}(\sigma_2(\alpha,k))^2 - \mu_2(1-\alpha)^2}.$$

Hence the result.

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Department of Mathematics, Al al-Bayt University, P.O.Box: 130095 Mafraq, Jordan.
E-mail: bafrasin@yahoo.com