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NEW INEQUALITIES FOR HERMITE-HADAMARD AND SIMPSON TYPE WITH APPLICATIONS

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Abstract. In this paper, we obtain new bounds for the inequalities of Simpson and Hermite-Hadamard type for functions whose second derivatives absolute values are P-convex. Some applications for special means of real numbers are also given.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following inequality, known as the Hermite-Hadamard inequality for convex functions, holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Since the inequalities in (1.1) have been also known as Hadamard's inequalities. In this work, we shall call them the Hermite-Hadamard inequalities or H-H inequalities, for simplicity.

In recent years many authors have established several inequalities connected to H-H inequality. For recent results, refinements, counterparts, generalizations and new H-H and Simpson type inequalities see the papers [2], [4], [5], [8], [9], [11], [12] and [13].

The following inequality is well known in the literature as Simpson's inequality.

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^2. \tag{1.2}$$

In [7], S.S. Dragomir et.al., defined following new class of functions.

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Definition 1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $P-convex$ function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y).$$

$P(I)$ contain all nonnegative monotone convex and quasi convex functions.

In [1], Akdemir and Özdemir defined co-ordinaded P-*convex* functions and proved some inequalities and in [7], Dragomir *et al.*, proved following inequalities of Hadamard's type for $P-convex$ functions.

Theorem 1. *Let $f \in P(I)$, $a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2\left[f(a)+f(b)\right].$$

In [6], Dragomir and Pearce have studied this type of inequalities for twice differential function with bounded second derivative and have obtained the following:

Theorem 2. *Assume that $f : I \to R$ is continuous on I, twice differentiable on I° and there exist k, K such that $k \leq f'' \leq K$ on I. Then*

$$\frac{k}{3}\left(\frac{b-a}{2}\right)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{K}{3}\left(\frac{b-a}{2}\right)^2. \tag{1.3}$$

In [3], Cerone and Dragomir proved the following theorem:

Theorem 3. *Let $f : [a,b] \to R$ be a twice differentiable mapping and suppose that $\gamma \leq f'' \leq \Gamma$ for all $t \in (a,b)$. Then we have*

$$\frac{\gamma(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(b-a)^2}{24}. \tag{1.4}$$

In [10], Sarıkaya *et al.* established following Lemma for twice differentiable mappings:

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval, with $a < b$. If $f : I \to \mathbb{R}$ is a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. Then the following identity holds:*

$$(\lambda-1)f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx = (b-a)^2\int_0^1 k(t)f''(ta+(1-t)b)dt$$

where

$$k(t) = \begin{cases} \frac{1}{2}t(t-\lambda), & 0 \leq t \leq \frac{1}{2} \\ \\ \frac{1}{2}(1-t)(1-\lambda-t), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The main purpose of this paper is to point out new estimations of the inequalities (1.1) and (1.2) and to apply them in special means of the real numbers.

2. Main Results

Using Lemma 1, we can obtain the following general integral inequalities for $P-$convex functions.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o (I^o is the interior of I) and $a, b \in I$ with $a < b$. If $\left|f''\right|$ is $P-$convex function, $0 \le \lambda \le 1$, then the following inequality holds:*

$$\left|(\lambda-1) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx\right| \le \begin{cases} \frac{(b-a)^2}{24}\left(8\lambda^3-3\lambda+1\right)\left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}, & for\ 0 \le \lambda \le \frac{1}{2} \\ \frac{(b-a)^2}{24}(3\lambda-1)\left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}, & for\ \frac{1}{2} \le \lambda \le 1. \end{cases} \tag{2.1}$$

Proof. From Lemma 1, we have

$$\begin{aligned} &\left|(\lambda-1) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{(b-a)^2}{2}\left[\int_0^{\frac{1}{2}} |t(t-\lambda)|\left|f''(ta+(1-t)b)\right|dt \right. \\ &\left. + \int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|\left|f''(ta+(1-t)b)\right|dt\right]. \end{aligned} \tag{2.2}$$

We assume that $0 \le \lambda \le \frac{1}{2}$, then using the $P-$convexity of $\left|f''\right|$, we have

$$\begin{aligned} &\int_0^{\frac{1}{2}} |t(t-\lambda)|\left|f''(ta+(1-t)b)\right|dt \\ &= \int_0^{\lambda} t(\lambda-t)\left|f''(ta+(1-t)b)\right|dt + \int_{\lambda}^{\frac{1}{2}} t(t-\lambda)\left|f''(ta+(1-t)b)\right|dt \\ &\le \left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\left[\int_0^{\lambda} t(\lambda-t)dt + \int_{\lambda}^{\frac{1}{2}} t(t-\lambda)dt\right] \\ &= \left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\left(\frac{\lambda^3}{3}-\frac{\lambda}{8}+\frac{1}{24}\right). \end{aligned} \tag{2.3}$$

Similarly, we write

$$\begin{aligned} &\int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|\left|f''(ta+(1-t)b)\right|dt \\ &= \int_{\frac{1}{2}}^{1-\lambda} (1-t)(1-\lambda-t)\left|f''(ta+(1-t)b)\right|dt + \int_{1-\lambda}^1 (1-t)(t+\lambda-1)\left|f''(ta+(1-t)b)\right|dt \end{aligned}$$

$$\le \left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\left[\int_{\frac{1}{2}}^{1-\lambda}(1-t)(1-\lambda-t)\,dt+\int_{1-\lambda}^{1}(1-t)(t+\lambda-1)\,dt\right]$$
$$= \left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\left(\frac{2(1-\lambda)^3}{3}+\lambda(1-\lambda)^2+\frac{7\lambda}{8}-\frac{5}{8}\right). \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), we see that first inequality of (2.1) holds.

On the other hand, let $\frac{1}{2}\le\lambda\le 1$, then, from $P-$convexity of $\left|f''\right|$ we have

$$\int_0^{\frac{1}{2}}|t(t-\lambda)|\left|f''(ta+(1-t)b)\right|dt+\int_{\frac{1}{2}}^{1}|(1-t)(1-\lambda-t)|\left|f''(ta+(1-t)b)\right|dt$$
$$\le \left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\left[\int_0^{\frac{1}{2}}t(\lambda-t)\,dt+\int_{\frac{1}{2}}^{1}(1-t)(t+\lambda-1)\,dt\right]$$
$$= \left\{\left|f''(a)\right|+\left|f''(b)\right|\right\}\left(\frac{\lambda}{4}-\frac{1}{12}\right).$$

This is second inequality of (2.1). This completes the proof. □

Theorem 5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o and $a,b\in I$ with $a<b$. If $\left|f''\right|^q$ is $P-$convex function, $0\le\lambda\le 1$ and $q\ge 1$, then the following inequality holds:*

$$\left|(\lambda-1)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le \begin{cases} \frac{(b-a)^2}{48}\left(8\lambda^3-3\lambda+1\right)\left(\left\{\left|f''(a)\right|^q+\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}, & for\ 0\le\lambda\le\frac{1}{2} \\ \frac{(b-a)^2}{48}\left(3\lambda-1\right)\left(\left\{\left|f''(a)\right|^q+\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}, & for\ \frac{1}{2}\le\lambda\le 1 \end{cases} \tag{2.5}$$

Proof. From Lemma 1 and using well known power mean inequality, we get

$$\left|(\lambda-1)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le\frac{(b-a)^2}{2}\left[\int_0^{\frac{1}{2}}|t(t-\lambda)|\left|f''(ta+(1-t)b)\right|dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}|(1-t)(1-\lambda-t)|\left|f''(ta+(1-t)b)\right|dt\right]$$
$$\le\frac{(b-a)^2}{2}\left(\int_0^{\frac{1}{2}}|t(t-\lambda)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t(t-\lambda)|\left[\left|f''(ta+(1-t)b)\right|\right]^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}|(1-t)(1-\lambda-t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}|(1-t)(1-\lambda-t)|\left[\left|f''(ta+(1-t)b)\right|\right]^q dt\right)^{\frac{1}{q}}. \tag{2.6}$$

Let $0\le\lambda\le\frac{1}{2}$. Since $\left|f''\right|$ is $P-$convex on $[a,b]$, we write

$$\int_0^{\frac{1}{2}}|t(t-\lambda)|\left[\left|f''(ta+(1-t)b)\right|\right]^q dt$$

$$
\begin{aligned}
&= \int_0^{\lambda} t(\lambda - t)\left[\left|f''(ta+(1-t)b)\right|\right]^q dt + \int_{\lambda}^{\frac{1}{2}} t(t-\lambda)\left[\left|f''(ta+(1-t)b)\right|\right]^q dt \\
&\leq \left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\left[\int_0^{\lambda} t(\lambda-t)dt + \int_{\lambda}^{\frac{1}{2}} t(t-\lambda)dt\right] \\
&= \left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\left(\frac{\lambda^3}{3} - \frac{\lambda}{8} + \frac{1}{24}\right), \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} |(1-t)(1-\lambda-t)|\left[\left|f''(ta+(1-t)b)\right|\right]^q dt \\
&\quad= \int_{\frac{1}{2}}^{1-\lambda} (1-t)(1-\lambda-t)\left[\left|f''(ta+(1-t)b)\right|\right]^q dt \\
&\qquad+ \int_{1-\lambda}^{1} (1-t)(t+\lambda-1)\left[\left|f''(ta+(1-t)b)\right|\right]^q dt \\
&\quad\leq \left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\left[\int_{\frac{1}{2}}^{1-\lambda} (1-t)(1-\lambda-t)dt + \int_{1-\lambda}^{1} (1-t)(t+\lambda-1)dt\right] \\
&\quad= \left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\left(\frac{2(1-\lambda)^3}{3} + \lambda(1-\lambda)^2 + \frac{7\lambda}{8} - \frac{5}{8}\right), \qquad (2.8)
\end{aligned}
$$

$$
\int_0^{\frac{1}{2}} |t(t-\lambda)|\,dt = \int_0^{\lambda} t(\lambda-t)dt + \int_{\lambda}^{\frac{1}{2}} t(t-\lambda)dt = \frac{\lambda^3}{3} + \frac{1-3\lambda}{24} \qquad (2.9)
$$

and

$$
\begin{aligned}
\int_{\frac{1}{2}}^{1} |(1-t)(1-\lambda-t)|\,dt &= \int_{\frac{1}{2}}^{1-\lambda} (1-t)(1-\lambda-t)dt + \int_{1-\lambda}^{1} (1-t)(t+\lambda-1)dt \\
&= \frac{\lambda^3}{3} + \frac{1-3\lambda}{24}. \qquad (2.10)
\end{aligned}
$$

Thus, using (2.7)-(2.10) in (2.6), we obtain the first inequality of (2.5).

Now, let $\frac{1}{2} \leq \lambda \leq 1$, then, using the $P-$convexity of $\left|f''\right|^q$, we have

$$
\begin{aligned}
\int_0^{\frac{1}{2}} |t(t-\lambda)|\left[\left|f''(ta+(1-t)b)\right|\right]^q dt &= \int_0^{\frac{1}{2}} t(\lambda-t)\left[\left|f''(ta+(1-t)b)\right|\right]^q dt \\
&\leq \int_0^{\frac{1}{2}} t(\lambda-t)\left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}dt \\
&= \left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\left(\frac{\lambda}{8} - \frac{1}{24}\right), \qquad (2.11)
\end{aligned}
$$

similarly,

$$
\begin{aligned}
\int_{\frac{1}{2}}^{1} |(1-t)(1-\lambda-t)|\left[\left|f''(ta+(1-t)b)\right|\right]^q dt &= \int_{\frac{1}{2}}^{1} (1-t)(t+\lambda-1)\left[\left|f''(ta+(1-t)b)\right|\right]^q dt \\
&\leq \int_{\frac{1}{2}}^{1} (1-t)(t+\lambda-1)\left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}dt
\end{aligned}
$$

$$= \left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \left(\frac{\lambda}{8} - \frac{1}{24} \right). \tag{2.12}$$

We also have

$$\int_0^{\frac{1}{2}} |t(t-\lambda)|\, dt = \int_{\frac{1}{2}}^1 |(1-t)(1-\lambda-t)|\, dt = \frac{3\lambda - 1}{24}. \tag{2.13}$$

Therefore, if we use the (2.11), (2.12) and (2.13) in (2.6), we obtain the second inequality of (2.5). This completes the proof. □

Corollary 1. *In Theorem 5, if we choose $\lambda = 0$, we obtain*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \le \frac{(b-a)^2}{48} \left(\left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right)^{\frac{1}{q}} \tag{2.14}$$

which similar to the left hand side of H-H inequality.

Corollary 2. *In Theorem 5 we choose $\lambda = 1$, we obtain*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{(b-a)^2}{24} \left(\left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right)^{\frac{1}{q}} \tag{2.15}$$

which similar to the right hand side of H-H inequality.

Corollary 3. *In Theorem 5, if we choose $\lambda = \frac{1}{3}$, we obtain*

$$\left| \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{(b-a)^2}{162} \left(\left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right)^{\frac{1}{q}}$$

which similar to the Simpson inequality.

Furthermore if f'' is bounded on $I = [a,b]$ then we have the following corollary:

Corollary 4. *In Corollary 1, if $\left| f'' \right| \le M$, $M > 0$, then we have*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \le M \frac{(b-a)^2}{48} 2^{\frac{1}{q}}.$$

Since $2^{\frac{1}{q}} \le 2$ for $q \ge 1$, we obtain

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2} \right) \right| \le M \frac{(b-a)^2}{24}$$

which is (1.4) inequality.

Corollary 5. *In Corollary 2, if $\left| f'' \right| \le M$, $M > 0$, then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le M \frac{(b-a)^2}{24} 2^{\frac{1}{q}}.$$

Since $2^{\frac{1}{q}} \le 2$ for $q \ge 1$, we obtain

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le M \frac{(b-a)^2}{12}$$

which is (1.3) inequality.

Corollary 6. *In Corollary 3, if* $\left|f''\right| \leq M$, $M > 0$, *then we have*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq M\frac{(b-a)^2}{162}2^{\frac{1}{q}}.$$

Since $2^{\frac{1}{q}} \leq 2$ *for* $q \geq 1$, *we obtain*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq M\frac{(b-a)^2}{81}.$$

3. Applications to Special Means

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

1. *Arithmetic mean*:
$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}^+.$$

2. *Logarithmic mean*:
$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha|-\ln\left|\beta\right|}, \quad |\alpha| \neq \left|\beta\right|,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}^+.$$

3. *Generalized log − mean*:
$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1,0\},\ \alpha,\beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 6. *Let* $a, b \in \mathbb{R}$, $0 < a < b$ *and* $n \in \mathbb{Z}$, $|n(n-1)| \geq 3$, *then, for all* $q \geq 1$, *the following inequality holds:*

$$\left|L_n^n(a,b)-A^n(a,b)\right| \leq |n(n-1)|\,\frac{(b-a)^2}{48}\left(\left\{a^{q(n-2)}+b^{q(n-2)}\right\}\right)^{\frac{1}{q}}.$$

Proof. The proof is obvious from Corollary 4 applied to the P-convex mapping $f(x) = x^n$, $x \in [a,b]$, $n \in \mathbb{Z}$. □

Proposition 7. *Let* $a, b \in \mathbb{R}$, $0 < a < b$ *and* $n \in \mathbb{Z}$, $|n(n-1)| \geq 3$, *then, for all* $q \geq 1$, *the following inequality holds:*

$$\left|A(a^n,b^n)-L_n^n(a,b)\right| \leq |n(n-1)|\,\frac{(b-a)^2}{24}\left(\left\{a^{q(n-2)}+b^{q(n-2)}\right\}\right)^{\frac{1}{q}}.$$

Proof. The proof is obvious from Corollary 6 applied to the P-convex mapping $f(x)=x^n$, $x\in[a,b]$, $n\in\mathbb{Z}$. □

Proposition 8. *Let $a,b\in\mathbb{R}$, $0<a<b$ and $n\in\mathbb{Z}$, $|n(n-1)|\geq 3$, then, for all $q\geq 1$, the following inequality holds:*

$$\left|\frac{1}{3}A(a^n,b^n)+\frac{2}{3}A^n(a,b)-L_n^n(a,b)\right|\leq |n(n-1)|\frac{(b-a)^2}{162}\left(\left\{a^{q(n-2)}+b^{q(n-2)}\right\}\right)^{\frac{1}{q}}.$$

Proof. The proof is obvious from Corollary 8 applied to the P-convex mapping $f(x)=x^n$, $x\in[a,b]$, $n\in\mathbb{Z}$. □

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