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AN INTEGRAL INVOLVING GENERAL POLYNOMIALS AND THE *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

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Abstract. The aim of this paper is to derive an integral pertaining to a product of Fox's H-function [1], general polynomials given by Srivastava [6, p.185, Eq.(7)] and H-function of several complex variables given by Srivastava and Panda [7, p.271, Eq. (4.1)] with general arguments of quadratic nature. The integral thus obtained is believed to be one of the most general integral established so far. The findings of this paper are sufficiently general in nature and are capable of yielding numerous (known or new) results involving classical orthogonal polynomials hitherto scattered in the literature.

1. Introduction

The series representation of Fox's H-function [1]:

$$H_{P,Q}^{M,N}\left[z\,\middle|\,\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right]=\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G}{G!F_g}\Phi(\eta_G)z^{\eta_G}, \tag{1.1}$$

where

$$\Phi(\eta_G)=\frac{\prod_{j=1 j\neq g}^{M}\Gamma(f_j-F_j\eta_G)\prod_{j=1}^{N}\Gamma(1-e_j+E_j\eta_G)}{\prod_{j=M+1}^{Q}\Gamma(1-f_j+F_j\eta_G)\prod_{j=N+1}^{P}\Gamma(e_j-E_j\eta_G)}$$

and

$$\eta_G=\frac{(f_g+G)}{F_g}.$$

The H-function of several complex variables is defined by Srivastava and Panda [7] as:

$$H[z_1,\dots,z_r]$$
$$=H_{A,C[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,\lambda:(u',v');\dots;(u^{(r)},v^{(r)})}\left[\begin{matrix}[(a):\theta';\dots;\theta^{(r)}]:[(b'):\Phi'];\dots;[(b^{(r)}):\Phi^{(r)}];\\ [(c):\Psi';\dots;\Psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix}z_1,\dots,z_r\right] \tag{1.2}$$

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The H-function of several complex variables in (1.2) converges absolutely if

$$|\arg(z_i)| < \frac{1}{2}\pi T_i, \tag{1.3}$$

where

$$T_i = -\sum_{j=1+\lambda}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \Phi_j^{(i)} - \sum_{j=1+v^{(i)}}^{B^{(i)}} \Phi_j^{(i)} - \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=1+u^{(i)}}^{D^{(i)}} \delta_j^{(i)} > 0,$$
$$\forall i \in (1, \ldots, r) \tag{1.4}$$

Srivastava has defined and introduced the general polynomials ([6], p.185, eq.(7)]

$$S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}[w_1,\ldots,w_r] = \sum_{\alpha_1=0}^{[N_1/M_1]} \cdots \sum_{\alpha_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!} \cdots \frac{(-N_s)_{M_s\alpha_s}}{\alpha_s!}$$
$$\cdot A[N_1,\alpha_1;\ldots;N_s,\alpha_s] w_1^{\alpha_1} \cdots w_s^{\alpha_s}, \tag{1.5}$$

where $N_i = 0, 1, 2, \ldots, \forall i = (1, \ldots, s)$; $M_1, \ldots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, \alpha_1; \ldots; N_s, \alpha_s]$ are arbitrary constants, real or complex.

2. The Main Integral

The following integral has been derived in this section:

$$\int_0^\infty t^{1-\alpha}(a+bt+ct^2)^{\alpha-3/2} H_{P,Q}^{M,N}\left[\left(\frac{t}{a+bt+ct^2}\right)^\sigma \middle| \begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right]$$
$$\cdot S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\left[w_1\left(\frac{t}{a+bt+ct^2}\right)^{n_1},\ldots,w_s\left(\frac{t}{a+bt+ct^2}\right)^{n_s}\right]$$
$$\cdot H\left[z_1\left(\frac{t}{a+bt+ct^2}\right)^{\sigma_1},\ldots,z_r\left(\frac{t}{a+bt+ct^2}\right)^{\sigma_r}\right] dt$$
$$= \sqrt{\frac{\pi}{c}} \sum_{G=0}^{\infty}\sum_{g=1}^{M}\sum_{\alpha_1=0}^{[N_1/M_1]} \cdots \sum_{\alpha_s=0}^{[N_s/M_s]} \frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_s)_{M_s\alpha_s}}{\alpha_s!}\Phi(\eta_G)$$
$$\cdot A[N_1,\alpha_1;\ldots;N_s,\alpha_s] w_1^{\alpha_1}\cdots w_s^{\alpha_s}(b+2\sqrt{ca})^{\alpha-\sigma\eta_G-\sum_{i=1}^{S} n_i\alpha_i - 1}$$
$$\cdot H_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:(u',v');\ldots;(u^{(r)},v^{(r)})}\left[\begin{matrix} z_1(b+2\sqrt{ca})^{-\sigma_1}\\ M \\ z_r(b+2\sqrt{ca})^{-\sigma_r}\end{matrix}\middle| \begin{matrix}[\alpha-\sigma\eta_G-\sum_{i=1}^{s} n_i\alpha_i:\sigma_1;\ldots;\sigma_r],\ [(c):\Phi';\ldots;\Phi^{(r)}],\\ [(a):\theta';\ldots;\theta^{(r)}] \quad :[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}]\\ [\alpha-\sigma\eta_G-\sum_{i=1}^{s} n_i\alpha_i-\frac{1}{2}:\sigma_1;\ldots;\sigma_r]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right] \tag{2.1}$$

provided that $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$, $c > 0$ and

$$\sigma \min\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right] + \sum_{i'=1}^{r} \sigma_{i'} \min\left[\mathrm{Re}\left(\frac{d_{j'}^{(i')}}{\delta_{j'}^{(i')}}\right)\right] > \alpha - 2,\ j = 1,\ldots,M \text{ and } j' = 1,\ldots,u^{(i')}.$$

Proof. In order to prove (2.1), we first express the Fox's H-function and a general polynomials in the form of series and the H-function of several complex variables in terms of Mellin-Barnes contour integrals. Now interchanging the order of summations and integrations which is permissible under the stated conditions, we obtain

$$\sum_{G=0}^{\infty}\sum_{g=1}^{M}\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_s=0}^{[N_s/M_s]} \frac{(-1)^G(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_1)_{M_s\alpha_s}}{\alpha_s!}\Phi(\eta_G)A[N_1,\alpha_1;\ldots;N_s,\alpha_s]$$
$$\cdot w_1^{\alpha_1}\cdots w_s^{\alpha_s}\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\Psi(\xi_1,\ldots,\xi_r)\Phi_1(\xi_1)\cdots\Phi_r(\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}$$
$$\cdot\Big\{\int_0^{\infty} t^{1-(\alpha-\sigma\eta_G-\sum_{i=1}^{s}n_i\alpha_i-\sigma_1\xi_1-\cdots-\sigma_r\xi_r)}$$
$$\cdot(a+bt+ct^2)^{(\alpha-\sigma\eta_G-\sum_{i=1}^{s}n_i\alpha_i-\sigma_1\xi_1-\cdots-\sigma_r\xi_r)-\frac{3}{2}}dt\Big\}d\xi_1\cdots d\xi_r, \tag{2.2}$$

Evaluating the above t-integral with the help of a known theorem (Saxena [5]) and reinterpreting the result thus obtained in terms of H-function of r-variables, we arrive at the desired result.

3. Particular Cases

I. Taking $\lambda = A$, $u^{(i)} = 1$, $v^{(i)} = B^{(i)}$ and $D^{(i)} = D^{(i)} + 1$, $\forall i \in (1,\ldots,r)$ the result in (2.1) reduces to the following integral transformation:

$$\int_0^{\infty} t^{1-\alpha}(a+bt+ct^2)^{\alpha-3/2}H_{P,Q}^{M,N}\left[\left(\frac{t}{a+bt+ct^2}\right)^{\sigma}\middle|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right]$$
$$\cdot S_{N_1,\ldots,N_s}^{M_1,\ldots,M_s}\left[w_1\left(\frac{t}{a+bt+ct^2}\right)^{n_1},\ldots,w_s\left(\frac{t}{a+bt+ct^2}\right)^{n_s}\right]$$
$$\cdot F_{C:D';\ldots;D^{(r)}}^{A:B';\ldots;B^{(r)}}\left[-z_1\left(\frac{t}{a+bt+ct^2}\right)^{\sigma_1},\ldots,-z_r\left(\frac{t}{a+bt+ct^2}\right)^{\sigma_s}\middle|\right.$$
$$\left.\begin{matrix}[1-(a):\theta';\ldots;\theta^{(r)}]\ :[1-(b'):\Phi'];\ldots;[1-(b^{(r)}):\Phi^{(r)}]\\ [1-(c):\Psi';\ldots;\Psi^{(r)}]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}]\end{matrix}\right]dt$$
$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{M}\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_s=0}^{[N_s/M_s]}\frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_s)_{M_s,\alpha_s}}{\alpha_s!}\Phi(\eta_G)$$

$$\cdot A[N_1,\alpha_1;\dots;N_s,\alpha_s]w_1^{\alpha_1}\cdots w_s^{\alpha_s}(b+2\sqrt{ca})^{\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i-1}$$
$$\frac{\Gamma(1-\alpha+\sigma\eta_G+\sum_{i=1}^{S}n_i\alpha_i)}{\Gamma(\frac{3}{2}-\alpha+\sigma\eta_G+\sum_{i=1}^{S}n_i\alpha_i)}$$
$$\cdot F^{A+1:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}\left[-z_1(b+2\sqrt{ca})^{-\sigma_1},\dots,-z_r(b+2\sqrt{ca})^{-\sigma_r}\right|$$
$$[1-\alpha+\sigma\eta_G+\sum_{i=1}^{s}n_i\alpha_i:\sigma_1;\dots;\sigma_r],[1-(a):\theta';\dots;\theta^{(r)}]$$
$$[1-(c):\Psi';\dots;\Psi^{(r)}],\left[\frac{3}{2}-\alpha+\sigma\eta_G+\sum_{i=1}^{s}n_i\alpha_i:\sigma_1;\dots;\sigma_r\right]$$
$$\left.\begin{array}{l}:[1-(b'):\Phi'];\dots;[1-(b^{(r)}):\Phi^{(r)}]\\ :[1-(d'):\delta'];\dots;[1-(d^{(r)});\delta^{(r)}]\end{array}\right], \tag{3.1}$$

provided that $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $c>0$, the series on the right side exists.

II. Taking $\theta',\dots,\theta^{(r)}=\Phi',\dots,\Phi^{(r)}=\Psi',\dots,\Psi^{(r)}=\delta',\dots,\delta^{(r)}=\sigma_1,\dots,\sigma_r=\alpha',\dots,\alpha^{(r)}$ in (2.1), we get the following integral transformation:

$$\int_0^\infty t^{1-\alpha}(a+bt+ct^2)^{\alpha-3/2}H^{M,N}_{P,Q}\left[\left(\frac{t}{a+bt+ct^2}\right)^{\sigma}\middle|\begin{array}{l}(e_p,E_p)\\(f_Q,F_Q)\end{array}\right]$$
$$\cdot S^{M_1,\dots,M_s}_{N_1,\dots,N_S}\left[w_1\left(\frac{t}{a+bt+ct^2}\right)^{n_1},\dots,w_s\left(\frac{t}{a+bt+ct^2}\right)^{n_S}\right]$$
$$\cdot G^{0,\lambda:(u',v');\dots;u^{(r)},v^{(r)})}_{A:C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[z_1^{1/\alpha'}\left(\frac{t}{a+bt+ct^2}\right),\dots,z_r^{1/\alpha^r}\left(\frac{t}{a+bt+ct^2}\right),\right|$$
$$\left.\begin{array}{l}(a):(b');\dots;(b^{(r)})\\(c):(d');\dots;(d^{(r)})\end{array}\right]dt$$
$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{M}\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_S=0}^{[N_s/M_s]}\frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_s)_{M_s,\alpha_s}}{\alpha_s!}\Phi(\eta_G)$$
$$\cdot A[N_1,\alpha_1;\dots;N_S,\alpha_S]w_1^{\alpha_1}\cdots w_S^{\alpha_S}(b+2\sqrt{ca})^{\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i-1}$$
$$\cdot G^{0,\lambda+1:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+1:C+1;[B',D'];\dots;[B^{(r)},D^{(r)}]}\left[z_1^{1/\alpha'}(b+2\sqrt{ca})^{-1},\dots,z_r^{1/\alpha^r}(b+2\sqrt{ca})^{-1}\right|$$
$$\left.\begin{array}{l}[\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i],(a):(b');\dots;(b^{(r)})\\(c),[\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i-\frac{1}{2}]:(d');\dots;(d^{(r)})\end{array}\right], \tag{3.2}$$

provided that $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $c>0$; $\alpha^{(i)}>0$ $(i=1,\dots,r)$, $2(u^{(i)}+v^{(i)})>(A+C+B^{(i)}+D^{(i)})$,

$$|\arg(z_i)|<\left[u^{(i)}+v^{(i)}-\frac{A}{2}-\frac{C}{2}-\frac{B^{(i)}}{2}-\frac{D^{(i)}}{2}\right]\pi \text{ and}$$

$$\sigma\left\{\min_{1\le j\le M}[\mathrm{Re}(f_j/F_j)]\right\}+\sum_{i=1}^{r}\left\{\min_{1\le j\le u^{(i)}}[\mathrm{Re}(d_j^{(i)})]\right\}>\alpha-2.$$

III. When $\lambda=A=C=0$ in (2.1), we have the following transformation:

$$\int_0^\infty t^{1-\alpha}(a+bt+ct^2)^{\alpha-3/2}H_{P,Q}^{M,N}\left[\left(\frac{t}{a+bt+ct^2}\right)^{\sigma}\middle|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right]$$
$$\cdot S_{N_1,\dots,N_S}^{M_1,\dots,M_s}\left[w_1\left(\frac{t}{a+bt+ct^2}\right)^{n_1},\dots,w_s\left(\frac{t}{a+bt+ct^2}\right)^{n_S}\right]$$
$$\cdot\prod_{i=1}^{r}H_{B^{(i)},D^{(i)}}^{u^{(i)},v^{(i)}}\left[z_i\left(\frac{t}{a+bt+ct^2}\right)^{\sigma_1}\middle|\begin{matrix}[(b^{(i)}):\Phi^{(i)}]\\ [d^{(i)}:\delta^{(i)}]\end{matrix}\right]dt$$
$$=\sqrt{\frac{\pi}{c}}\sum_{G=0}^{\infty}\sum_{g=1}^{M}\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_S=0}^{[N_s/M_s]}\frac{(-1)^G}{G!F_g}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_s)_{M_s,\alpha_s}}{\alpha_s!}\Phi(\eta_G)$$
$$\cdot A[N_1,\alpha_1;\dots;N_S,\alpha_S]w_1^{\alpha_1}\cdots w_S^{\alpha_S}(b+2\sqrt{ca})^{\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i-1}$$
$$\cdot H_{1,1;[B',D'];\dots;[B^{(r)},D^{(r)}]}^{0,1:(u',v');\dots;(u^{(r)},v^{(r)})}\left[\begin{matrix}z_1(b+2\sqrt{ca})^{-\sigma_1}\\ \vdots\\ z_r(b+2\sqrt{ca})^{-\sigma_r}\end{matrix}\middle|\begin{matrix}[\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i:\sigma_1;\dots;\sigma_r]:[(b'):\Phi'];\dots;[(b^{(r)}):\Phi^{(r)}]\\ [\alpha-\sigma\eta_G-\sum_{i=1}^{S}n_i\alpha_i-\frac{1}{2}:\sigma_1;\dots;\sigma_r]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\right],\qquad(3.3)$$

valid under the same conditions as obtainable from (2.1)

IV. Replacing $N_1,\dots,N_s$ by N the result in (2.1) reduces to the known result given in [2], after a little simplification.

V. Taking $N_i\to 0$, $(i=1,\dots,s)$, $a=0$, $c=1$, the result in (2.1) reduces to the known result after a slight simplification obtained by Goyal and Mathur [3].

VI. If $r=1$ and M_i, $N_i\to 0$ $(i=2,\dots,s)$, the result in (2.1) reduces to the known result with a slight modification recently derived by Gupta and Jain [4].

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