A NOTE ON AN OPEN PROBLEM

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Abstract. The function $\frac{\Gamma(x+1)}{(x+\beta)^{\alpha}}$ is logarithmically completely monotonic on $(0,\infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$, and is logarithmically completely monotonic in $(-1,0)$ for $0 < \alpha \leq \frac{2\beta}{1+2\beta}$ and $\beta > 1$. This give an answer to an open problem proposed by Feng Qi.

1. Introduction

The classical gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \quad (x > 0) \tag{1}$$

is one of the most important functions in analysis and its applications. The history and development of this function are described in detail [2]. The psi or digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed [6, p.16] as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \, dt, \tag{2}$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \, dt \tag{3}$$

for $x > 0$ and $k = 1, 2, \ldots$, where $\gamma = 0.57721566490153286 \ldots$ is the Euler-Mascheroni constant.

We recall that a function $f : (0,\infty) \rightarrow \mathbb{R}$ is said to be completely monotonic if $f$ has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{4}$$

for $x > 0$ and $n = 0, 1, 2, \ldots$. If $f$ is nonconstant and completely monotonic, then the inequality (4) is strict, see [3]. Let $\mathcal{C}$ denote the set of completely monotonic functions.

A function $f$ is said to be logarithmically completely monotonic on $(0,\infty)$ if $f$ is positive and, for all $n \in \mathbb{N},$

$$0 \leq (-1)^n [\log f(x)]^{(n)} < \infty, \tag{5}$$
see[1, 7]. If inequality (5) is strict for all \( x \in (0, \infty) \) and for all \( n \geq 1 \), then \( f \) is said to be strictly logarithmically completely monotonic. Let \( \mathcal{L} \) on \((0, \infty)\) stand for the set of logarithmically completely monotonic functions.

The notion that logarithmically completely monotonic function was posed explicitly in [8] and published formally in [7] and a much useful and meaningful relation \( \mathcal{L} \subset \mathcal{C} \) between the completely monotonic functions and the logarithmically completely monotonic functions was proved in[7, 8].

In [5], H. Minc and L. Sathre proved that, if \( n \) is a positive integer and \( \phi(n) = (n!)^{\frac{1}{n}} \), then

\[
1 < \frac{\phi(n + 1)}{\phi(n)} < \frac{n + 1}{n},
\]

which can be rearranged as

\[
[\Gamma(n + 1)]^{1/n} < [\Gamma(n + 2)]^{1/(n+1)}
\]

and

\[
\frac{[\Gamma(n + 1)]^{1/n}}{n} > \frac{[\Gamma(n + 2)]^{1/(n+1)}}{n+1},
\]

since \( \Gamma(n + 1) = n! \).

In [4], the following monotonicity results for the Gamma function were established. The function \( \Gamma(1 + \frac{1}{x})^{x} \) decreases with \( x > 0 \) and \( x \Gamma(1 + \frac{1}{x})^{x} \) increases with \( x > 0 \), which recover the inequalities in (6) which refer to integer values of \( n \). These are equivalent to the function \( [\Gamma(1 + x)]^{1/x} \) being increasing and \( \frac{\Gamma(1+x)^{1/x}}{x} \) being decreasing on \((0, \infty)\), respectively. In addition, it was proved that the function \( x^{1-\gamma} \left[ \Gamma(1 + \frac{1}{x})^{x} \right] \) decreases for \( 0 < x < 1 \), which is equivalent to \( \frac{\Gamma(1+x)^{1/x}}{x^{1-\gamma}} \) being increasing on \((1, \infty)\).

In [9], Qi and Chen showed that the function \( \frac{\Gamma(x+1)}{x+1} \) is strictly decreasing and strictly logarithmically convex in \((0, \infty)\), and the function \( \frac{\Gamma(x+1)^{1/x}}{\sqrt{x+1}} \) is strictly increasing and strictly logarithmically concave in \((0, \infty)\). Using the monotonicity of above functions, Qi and Chen presented the following double inequality

\[
\frac{x + 1}{y + 1} < \frac{\Gamma(x + 1)^{1/x}}{\Gamma(y + 1)^{1/y}} < \sqrt{\frac{x + 1}{y + 1}}
\]

for \( 0 < x < y \), see Corollary 1 of [9].

In [8], Qi and Guo proposed an open problem

**Open Problem 1.** Find conditions about \( \alpha \) and \( \beta \) such that the ratio

\[
F(x) = \frac{[\Gamma(x + 1)]^{\frac{1}{x}}}{(x + \beta)^{\alpha}}
\]

is completely (absolutely, regularly) monotonic (convex) with \( x > -1 \).
In this paper, we give an answer to this problem and establish new inequalities.

**Theorem 1.** The function $F(x)$ defined by (7) is strictly logarithmically completely monotonic in $(0, \infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$. Moreover, the function $F(x)$ is strictly completely monotonic in $(0, \infty)$ for $\alpha \geq 1$ and $0 \leq \beta \leq 1$.

**Proof.** Taking the logarithm of $F(x)$ defined by (7),

$$\log F(x) = \frac{\log \Gamma(x + 1)}{x} - \alpha \log(x + \beta)$$

\[ \triangleq g(x) - \alpha \log(x + \beta). \tag{8} \]

Using Leibniz’ rule

\[ [u(x)v(x)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)}(x)v^{(n-k)}(x), \tag{9} \]

we have

\[ g^{(n)}(x) = \frac{1}{x^{n+1}} \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x + 1)}{k!} \triangleq h_n(x) \frac{x^n}{x^{n+1}}. \tag{10} \]

\[ h'_n(x) = x^n \psi^{(n)}(x + 1) \begin{cases} > 0, & \text{if } n \text{ is odd and } x \in (0, \infty), \\ \leq 0, & \text{if in is odd and } x \in (-1, 0) \text{ and } n \text{ is even and } x \in (-1, \infty), \end{cases} \tag{11} \]

where $\psi^{(-1)}(x + 1) = \log \Gamma(x + 1)$ and $\psi^{(0)}(x + 1) = \psi(x + 1)$.

\[ (-1)^n \left( \log F(x) \right)^{(n)} = \frac{1}{x^{n+1}} \left[ (-1)^n h_n(x) + \frac{(n-1)! \alpha x^{n+1}}{(x+\beta)^n} \right] \]

\[ \triangleq \frac{v_{\alpha,\beta}(x)}{x^{n+1}} \]

Using the representations

\[ \frac{(n-1)!}{(x+1)^n} = \int_{0}^{\infty} t^{n-1} e^{-(x+1)t} \, dt, \quad x > 0, \quad n = 1, 2, \ldots, \tag{12} \]

and (3), we conclude

\[ v_{\alpha,\beta}(x) = (-1)^n x^n \psi^{(n)}(x + 1) + \frac{n! x^n \alpha \beta}{(x+\beta)^{n+1}} + \frac{(n-1)! x^n \alpha}{(x+\beta)^n} \]

\[ = x^n \int_{0}^{\infty} \left[ \alpha (e^t - 1) + \alpha \beta t (e^t - 1) - t e^{\beta t} \right] e^{-x \beta t} e^{-t} \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} \, dt \]

\[ \triangleq x^n \int_{0}^{\infty} \phi(t) \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} \, dt \tag{13} \]
where
\[
\phi(t) = \alpha \beta t (e^t - 1) - te^{\beta t} + \alpha (e^t - 1) = (\alpha - 1) t + \sum_{m=2}^{\infty} \frac{[\alpha + m\beta(\alpha - \beta^{m-2})] t^m}{m!}.
\]

If \(\alpha \geq 1\) and \(0 \leq \beta \leq 1\), then \(\phi(t) > 0\) and \(\nu_{\alpha,\beta}'(x) > 0\). Hence, \(\nu_{\alpha,\beta}(x) > \nu_{\alpha,\beta}(0) = 0\) and \((-1)^{n} (\log F(x))^{(n)} > 0\), and thus, the function \(F(x)\) is strictly logarithmically completely monotonic. The proof of Theorem 1 is complete.

**Corollary 1.** For \(\alpha \geq 1\) and \(0 \leq \beta \leq 1\),
\[
\frac{\Gamma(x+1)^{\frac{1}{x}}}{\Gamma(y+1)^{\frac{1}{y}}} > \left(\frac{x + \beta}{y + \beta}\right)^{\alpha}, \quad (14)
\]
in which \(0 < x < y\).

**Theorem 2.** The function \(F(x)\) defined by \((7)\) is strictly logarithmically completely monotonic in \((-1, 0)\) for \(0 < \alpha \leq \frac{2\beta}{1+2\beta}\) and \(\beta > 1\). Moreover, the function \(F(x)\) is strictly completely monotonic in \((-1, 0)\) for \(0 < \alpha \leq \frac{2\beta}{1+2\beta}\) and \(\beta > 1\).

**Proof.** By \((13)\),
\[
\phi(t) = \alpha \beta t (e^t - 1) - te^{\beta t} + \alpha (e^t - 1) \\
\phi(0) = 0 \\
\phi'(t) = e^t (\alpha + \alpha \beta + \alpha \beta t) - \alpha \beta - e^{\beta t} (1 + \beta t) \\
\phi'(0) = \alpha - 1 \\
\phi''(t) = e^t \left[\alpha + 2\alpha \beta + \alpha \beta t - \beta e^{(\beta-1)t} (2 + \beta t)\right] \\
\triangleq e^t u(t) \\
\phi''(0) = 0 \\
\phi''(t) = e^{(\beta-1)t} \left[\alpha + 2\alpha \beta + \alpha \beta t - \beta e^{(\beta-1)t} (2 + \beta t)\right]
\]

If \(0 < \alpha \leq \frac{2\beta}{1+2\beta}\) and \(\beta > 1\), then \(u''(t) < 0\) and \(u'(t)\) is strictly decreasing. So \(u'(t) < u'(0) < 0\) and \(u(t)\) is strictly decreasing. Hence, \(u(t) < u(0) < 0\) and \(\phi''(t) < 0\). Since \(0 < \alpha \leq \frac{2\beta}{1+2\beta}\), we have \(\phi'(t) < \phi'(0) < 0\). So we conclude that \(\phi(t) < \phi(0) = 0\).
If $n$ is odd, then $v_{a,\beta}'(x) > 0$ on $(-1, 0)$, and then, $v_{a,\beta}(x) > v_{a,\beta}(0) = 0$ and $(-1)^n (\log F(x))^{(n)} > 0$. If $n$ is even, then $v_{a,\beta}'(x) < 0$ on $(-1, 0)$, and then, $v_{a,\beta}(x) < v_{a,\beta}(0) = 0$ and $(-1)^n (\log F(x))^{(n)} > 0$ on $(-1, 0)$.

This means that the function $F(x)$ is strictly logarithmically completely monotonic on $(-1, 0)$. The proof of Theorem 2 is complete.

**Corollary 2.** For $0 < \alpha \leq \frac{2\beta}{1+2\beta}$ and $\beta > 1$, 

$$\frac{\Gamma(x+1)^\frac{1}{\beta}}{\Gamma(y+1)^\frac{1}{\beta}} > \left(\frac{x+\beta}{y+\beta}\right)^{\alpha},$$

in which $-1 < x < y < 0$.

Motivated by the open problem, we established a new function 

$$G(x) = \frac{[\Gamma(x+\alpha)]^{\frac{1}{\gamma}}}{(x+\beta)^\gamma}$$

in which $\alpha, \beta, \gamma$ are nonnegative. Our Theorem 3 considers its logarithmically completely monotonicity.

**Theorem 3.** The function $G(x)$ defined by (16) is strictly logarithmically completely monotonic in $(0, \infty)$ for $\alpha \in (0, 1] \cup [2, \infty)$, $\alpha - 1 \leq \beta \leq \alpha$ and $\gamma \geq \max\left\{\frac{1}{\beta}, 1\right\}$. Moreover, the function $G(x)$ is strictly completely monotonic in $(0, \infty)$ for $\alpha \in (0, 1] \cup [2, \infty)$, $\alpha - 1 \leq \beta \leq \alpha$ and $\gamma \geq \max\left\{\frac{1}{\beta}, 1\right\}$.

**Proof.** Using (9), we obtain

$$(\log G(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} \left[\log \Gamma(x+\alpha)\right]^{(k)} - \frac{(-1)^{n-1} \gamma (n-1)!}{(x+\beta)^n}$$

$$= \left(\frac{1}{x}\right)^{(n)} \log \Gamma(x+\alpha) + \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} \psi^{(k-1)}(x+\alpha) + \frac{(-1)^{n} \gamma (n-1)!}{(x+\beta)^n}$$

$$= \frac{(-1)^{n} n!}{x^{n+1}} \log \Gamma(x+\alpha) + \sum_{k=1}^{n} \frac{n!}{k!} (-1)^{n-k} \psi^{(k-1)}(x+\alpha) + \frac{(-1)^{n} \gamma (n-1)!}{(x+\beta)^n}$$

$$\triangleq (-1)^{n} \frac{1}{x^{n+1}} \delta(x),$$

and

$$\delta'(x) = x^n \left( (-1)^{n} \psi^{(n)}(x+\alpha) + \frac{n! \beta \gamma}{(x+\beta)^{n+1}} + \frac{(n-1)! \gamma}{(x+\beta)^n} \right).$$
Using (3) and (12) for $x > 0$ and $n \in \mathbb{N}$, we conclude
\[
\frac{1}{x^n} \delta'(x) = (-1)^n \psi^{(n)}(x + \alpha) + \frac{n! \beta \gamma}{(x + \beta)^{n+1}} + \frac{(n-1)! \gamma}{(x + \beta)^n}
\]
\[
= \int_0^\infty \left[ \gamma(e^t - 1) + \beta \gamma t(e^t - 1) - t e^{(\beta - \alpha + 1)t} \right] \left[ \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} \right] dt
\]
\[
\triangleq \int_0^\infty u(t) \frac{t^{n-1} e^{-(x+\beta)t}}{e^t - 1} dt,
\]
where
\[
u(t) = \beta \gamma t(e^t - 1) - t e^{(\beta - \alpha + 1)t} + \gamma(e^t - 1)
\]
\[
= (\gamma - 1) t + \sum_{m=2}^\infty \left\{ \gamma + m \left[ \beta \gamma - (\beta - \alpha + 1)^{m-1} \right] \right\} t^m / m!.
\]

If $\alpha - 1 \leq \beta \leq \alpha$ and $\gamma \geq \max \left\{ \frac{1}{\beta}, 1 \right\}$, then $\nu(t) > 0$ and $\delta'(x) > 0$. Notice that $\Gamma(\alpha) \geq 1$ for $\alpha \in (0, 1] \cup [2, \infty)$. Hence, $\delta(x) > \delta(0) = n! \log \Gamma(\alpha) \geq 0$ and $(-1)^n \left( \log G(x) \right)^{(n)} > 0$ in $(0, \infty)$, and thus, the function $G(x)$ is strictly logarithmically completely monotonic. The proof of Theorem 3 is complete. □

**Corollary 3.** For $\alpha \in (0, 1] \cup [2, \infty)$, $\alpha - 1 \leq \beta \leq \alpha$ and $\gamma \geq \max \left\{ \frac{1}{\beta}, 1 \right\}$,
\[
\frac{\Gamma(x+\alpha)}{\Gamma(y+\alpha)} > \frac{(x+\beta)^\gamma}{(y+\beta)^\gamma},
\]
in which $0 < x < y$.

**References**


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