OSCILLATION CRITERION FOR TWO-DIMENSIONAL
DYNAMIC SYSTEMS ON TIME SCALES

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Abstract. The purpose of this paper is to prove oscillation criterion for dynamic system

\[ u^\Delta = pv, \quad v^\Delta = -qu^\sigma, \]

where \( p > 0 \) and \( q \) are rd-continuous functions on a time scale such that \( \sup \mathbb{T} = \infty \) without explicit sign assumptions on \( q \) and also without restrictive conditions on the time scale \( \mathbb{T} \).

1. Introduction

We consider the linear dynamic system

\[ u^\Delta = pv, \quad v^\Delta = -qu^\sigma, \tag{1.1} \]

where \( p > 0 \) and \( q \) are rd-continuous functions on a time scale such that \( \sup \mathbb{T} = \infty \). A solution \((u(t), v(t))\) of system (1.1) is called oscillatory if both \( u(t) \) and \( v(t) \) are oscillatory functions, and otherwise it will be called nonoscillatory. System (1.1) is called oscillatory if its solutions are oscillatory.

In [6, 1], the following oscillation theorem is obtained.

**Theorem 1.1.** Assume that \( q(t) \geq 0 \) is rd-continuous function and

\[ \int_{t_0}^{\infty} p(t) \Delta t = \infty \quad \text{and} \quad \lim_{t \to \infty} \mu(t) \frac{p(t)}{\overline{P}(t)} = 0. \tag{1.2} \]

If there exists \( \lambda \in (0, 1) \) such that

\[ \int_{t_0}^{\infty} p^\lambda(t) q(t) \Delta t = \infty, \]

where

\[ \overline{P}(t) := \int_{t_0}^{t} p(s) \Delta s, \tag{1.3} \]

then system (1.1) is oscillatory.

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Recently, Baoguo [5] proved above theorem when \( q(t) \) is allowed to take on negative values and established the following theorem.

**Theorem 1.2.** Assume that \( \mathbb{T} \) satisfies condition (C) and (1.2). If there exists \( \lambda \in [0, 1) \) such that

\[
\int_{t_0}^{\infty} \overline{p}(\sigma(t)) q(t) \Delta t = \infty, \tag{1.4}
\]

where \( \overline{p}(t) \) is defined by (1.3), then system (1.1) is oscillatory.

To be precise, we say \( \mathbb{T} \) satisfies condition (C), that is there is an \( M > 0 \) such that \( \chi(t) \leq M \mu(t), \ t \in \mathbb{T} \), where \( \chi \) is the characteristic function of the set \( \hat{\mathbb{T}} = \{ t \in \mathbb{T} : \mu(t) > 0 \} \). We note that if \( \mathbb{T} \) satisfies condition (C), then the subset \( \tilde{\mathbb{T}} \) of \( \mathbb{T} \) defined by

\[
\tilde{\mathbb{T}} = \{ t \in \mathbb{T} : t > 0 \text{ is right-scattered or left-scattered} \},
\]

is necessarily countable and \( \tilde{\mathbb{T}} \subset \mathbb{T} \). Then, we can rewrite \( \tilde{\mathbb{T}} \) by

\[
\tilde{\mathbb{T}} = \{ t_i \in \mathbb{T} : 0 < t_1 < t_2 < ... < t_n < ... \},
\]

and so

\[
\mathbb{T} = \tilde{\mathbb{T}} \cup \bigcup_{n \in A} (t_{n-1}, t_n),
\]

where \( A \) is the set of all integers for which the real open interval \((t_{n-1}, t_n)\) is contained in \( \mathbb{T} \). There are several time scales do not satisfy condition (C), for example it is easy to see the time scale in the form

\[
\mathbb{T} = \bigcup_{k=1}^{\infty} T_k, \text{ where } T_k = \bigcup_{n=1}^{\infty} \left\{ k + \frac{n+1}{n} \right\},
\]

does not satisfy condition (C).

Therefore it will be of great interest to prove Theorems 1.1 and 1.2 without explicit sign assumption on \( q \) and also without condition (C) to the same dynamic system (1.1), so our work improves and generalizes those established in [6, 5].

2. Main results

Before stating the main results, we start with the following lemmas which will play an important role in the proofs of the main results.

**Lemma 2.1.** For all \( \lambda \in [0, 1) \) and for sufficiently large \( T \geq t_0 \), we have, for \( t \in [T, \infty)_{\mathbb{T}} \)

\[
\int_{T}^{t} p(s) \left[ \overline{p}(\sigma(s)) \right]^{\lambda-2} \Delta s \leq \frac{[\overline{p}(T)]^{\lambda-1}}{1-\lambda},
\]

where \( \overline{p}(t) \) is defined by (1.3).
Lemma 2.2. If \( y \) and \( x \) are differentiable on \( \mathbb{T} \) and \( x \neq 0 \) on \( \mathbb{T} \), then
\[
x^\Delta \left( \frac{y^2}{x} \right)^\Delta = (y^\Delta)^2 - \frac{x^\Delta y^\sigma}{\sqrt{1 + \mu x^\Delta x y^\Delta}}.
\]

Proof. From the quotient rule, we get
\[
x^\Delta \left( \frac{y^2}{x} \right)^\Delta = x^\Delta \left[ \frac{x(y^2)^\Delta - y^2 x^\Delta}{xx^\sigma} \right] = x^\Delta \left[ x(y^2)^\Delta - y^2 x^\Delta \right]
\]

Since \((y^2)^\Delta = (yy)^\Delta = y^\Delta (y^\sigma + y) = y^\Delta (2y + \mu y^\Delta)\), then
\[
x^\Delta \left( \frac{y^2}{x} \right)^\Delta = \frac{x^\Delta}{xx^\sigma} \left[ 2xy y^\Delta + \mu x (y^\Delta)^2 - x^\Delta y^2 \right]
\]
\[
= \frac{x^\Delta}{xx^\sigma} \left[ 2yy^\Delta (x^\sigma - \mu x^\Delta) + \mu (y^\Delta)^2 (x^\sigma - \mu x^\Delta) - x^\Delta y^2 \right]
\]
\[
= \frac{x^\Delta}{xx^\sigma} \left[ -x^\Delta (y^2 + 2\mu y y^\Delta + \mu^2 (y^\Delta)^2) + x^\sigma y^\Delta (y + \mu y^\Delta) + x^\sigma y^\sigma (y^\sigma - \mu y^\Delta) \right]
\]
\[
= \frac{x^\Delta}{xx^\sigma} \left[ -x^\Delta (y^\sigma)^2 + 2x^\sigma y^\Delta y^\sigma - \mu x^\sigma (y^\Delta)^2 \right]
\]
\[
= - \left( \frac{x^\Delta y^\sigma}{x} \right)^2 \frac{2}{x^\sigma} + \frac{2x^\Delta}{x} y^\Delta y^\sigma - \frac{\mu x^\Delta}{x} (y^\Delta)^2
\]
\[
= - \left( \frac{x^\Delta y^\sigma}{x} \right)^2 \frac{2x^\Delta}{x} y^\Delta y^\sigma - \frac{\mu x^\Delta}{x} (y^\Delta)^2
\]
\[
\begin{align*}
\frac{x^\Delta y^\Delta}{x} + \sqrt{1 + \mu \frac{x^\Delta}{x} y^\Delta} \right)^2 + (y^\Delta)^2,
\end{align*}
\]

since \(1 + \mu \frac{x^\Delta}{x} = \frac{x^\sigma}{x} \neq 0\). \(\blacksquare\)

From above lemmas and motivated by the proof of Theorem 3.1 in [5], we can prove Theorem 1.1 without condition (C).

**Theorem 2.3.** Assume that
\[
\int_{t_0}^{\infty} p(t) \Delta t = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{\mu(t) p(t)}{p(t)} = 0. \quad (2.1)
\]

If there exists \(\lambda \in [0, 1)\) such that
\[
\int_{t_0}^{\infty} \overline{P}^\lambda (\sigma(t)) q(t) \Delta t = \infty, \quad (2.2)
\]

where \(\overline{P}(t)\) is defined by (1.3), then system (1.1) is oscillatory.

**Proof.** Assume that \((u(t), v(t))\) is a nonoscillatory solution of system (1.1). We claim that \(u(t)\) is nonoscillatory. If not, we assume \(u(t)\) is oscillatory and \(v(t)\) is nonoscillatory. Without loss of generality, we let \(v(t) > 0\) on \([t_0, \infty)_\mathbb{I}\). In the view of the first equation of system (1.1), we have \(u^\Delta(t) > 0\) on \([t_0, \infty)_\mathbb{I}\). Thus \(u(t) > 0\) or \(u(t) < 0\) for all large \(t\), which is a contradiction. Thus \(u(t)\) is nonoscillatory and without loss of generality, we let \(u(t) > 0\) for \(t \geq T \geq t_0\).

Define, for \(\lambda \in [0, 1)\)
\[
w(t) := \frac{\overline{P}^\lambda(t) u^\Delta(t)}{p(t) u(t)}.
\]

Then, from the product rule, we get (suppressing arguments)
\[
w^\Delta = \left[ \frac{\overline{P}^\lambda}{u} \right]^\sigma \left( \frac{1}{p} u^\Delta \right)^\Delta + \left[ \frac{\overline{P}^\lambda}{u} \right]^\Delta \left( \frac{1}{p} u^\Delta \right) = -\left( \frac{\overline{P}^\lambda}{u} \right)^\sigma q + \frac{1}{p} \left[ u^\Delta \left( \frac{\overline{P}^{\lambda/2}}{u} \right)^2 \right]^\Delta.
\]

Now, by using Lemma 2.2 with replaced \(x\) by \(u\) and \(y\) by \(\overline{P}^{\lambda/2}\), we get
\[
w^\Delta = -\left( \frac{\overline{P}^\lambda}{u} \right)^\sigma q + \frac{1}{p} \left[ u^\Delta \left( \frac{\overline{P}^{\lambda/2}}{u} \right)^2 \right]^\Delta
\]
\[
= -\left( \frac{\overline{P}^\lambda}{u} \right)^\sigma q + \frac{1}{p} \left[ \left( \frac{\overline{P}^{\lambda/2}}{u} \right)^2 \right]^\Delta - \left[ \frac{u^\Delta \left( \frac{\overline{P}^{\lambda/2}}{u} \right)^2}{1 + \mu u^\Delta \overline{P}^{\lambda/2}} \right]^2
\]
Therefore,
\[
\int_T^t w^\Delta (s) \, \Delta s \leq - \int_T^t \overline{p}^\lambda (\sigma (s)) \, q (s) \, \Delta s + \int_T^t \left( \frac{\overline{p}^{\lambda / 2} (s)}{p (s)} \right)^2 \Delta s.
\] (2.3)

Using the Pötzsche chain rule ([3, Theorem 1.90]), we get
\[
\left( \overline{p}^{\lambda / 2} (t) \right)^\Delta = \frac{\lambda}{2} \int_0^1 [1 - (1 - h) \overline{p} (t) + h \overline{p} (\sigma (t))]^{\lambda / 2 - 1} \, dh \, p (t).
\] (2.4)

Since \( \lim_{t \to \infty} \frac{\mu (t) p (t)}{\overline{p} (t)} = 0 \), it implies that
\[
\lim_{t \to \infty} \frac{\overline{p} (\sigma (t))}{\overline{p} (t)} = \lim_{t \to \infty} \frac{\int_0^t p (s) \Delta s + \int_t^\sigma (t) p (s) \Delta s}{\int_0^t p (s) \Delta s} = 1 + \lim_{t \to \infty} \frac{\mu (t) p (t)}{\overline{p} (t)} = 1,
\]
so for given \( 0 < \epsilon < 1 \), there exists \( T_1 \) sufficiently large such that
\[
\overline{p} (t) \geq (1 - \epsilon) \overline{p} (\sigma (t)), \quad \text{for } t \geq T_1.
\]

Then, from (2.4), we have
\[
\left( \overline{p}^{\lambda / 2} (t) \right)^\Delta \leq \frac{\lambda}{2} p (t) [\overline{p} (\sigma (t))]^{\lambda / 2 - 1} \int_0^1 [1 - (1 - h) \epsilon]^{\lambda / 2 - 1} \, dh
\]
\[
= p (t) [\overline{p} (\sigma (t))]^{\lambda / 2 - 1} \frac{1}{\epsilon} \left[ 1 - (1 - \epsilon)^{\lambda / 2} \right]
\]
\[
= M p (t) [\overline{p} (\sigma (t))]^{\lambda / 2 - 1},
\] (2.5)

where \( M := \frac{1}{\epsilon} [1 - (1 - \epsilon)^{\lambda / 2}] \). From (2.3) and (2.5), we get
\[
w (t) - w (T) \leq - \int_T^t \overline{p}^\lambda (\sigma (s)) \, q (s) \, \Delta s + \int_T^t \left( \frac{\overline{p}^{\lambda / 2} (s)}{p (s)} \right)^2 \Delta s
\]
\[
\leq - \int_T^t \overline{p}^\lambda (\sigma (s)) \, q (s) \, \Delta s + M^2 \int_T^t p (s) [\overline{p} (\sigma (s))]^{\lambda / 2 - 2} \Delta s.
\]

Since
\[
\frac{1}{\lambda - 1} [\overline{p} (s)]^{\lambda - 1} = \int_0^1 [1 - (1 - h) \overline{p} (s) + h \overline{p} (\sigma (s))]^{\lambda - 2} \, dh \, p (s)
\]
\[
\geq \int_0^1 [1 - (1 - h) \overline{p} (\sigma (s)) + h \overline{p} (\sigma (s))]^{\lambda - 2} \, dh \, p (s)
\]
\[
= [\overline{p} (s)]^{\lambda - 2} p (s),
\]
which yields
\[
w(t) - w(T) \leq - \int_T^t \overline{p}^\lambda(\sigma(s)) q(s) \Delta s + M^2 \left( \frac{[\overline{p}(t)]^{\lambda-1}}{\lambda-1} - \frac{[\overline{p}(T)]^{\lambda-1}}{\lambda-1} \right).
\]

In view of condition (2.2), it follows from the last inequality that there exists a sufficiently large \( T_2 \geq T_1 \) such that
\[
u^\Delta(t) < 0, \quad \text{for } t \in [T_2, \infty) \cap I.
\]

By using [3, Theorem 4.61], there is a positive solution \( \tilde{u} \), called dominant solution such that
\[
\int_{T_2}^{\infty} \frac{p(t)}{\tilde{u}(t) \tilde{u}^\sigma(t)} \Delta t < \infty,
\]
and then, from (2.6), we get
\[
\tilde{u}^\Delta(t) < 0, \quad \text{for } t \in [T_2, \infty) \cap I.
\]

This implies
\[
\int_{T_2}^{\infty} p(t) \Delta t = \tilde{u}(T_2) \tilde{u}^\sigma(T_2) \int_{T_2}^{\infty} \frac{p(t)}{\tilde{u}(T_2) \tilde{u}^\sigma(T_2)} \Delta t \leq \tilde{u}(T_2) \tilde{u}^\sigma(T_2) \int_{T_2}^{\infty} \frac{p(t)}{\tilde{u}(t) \tilde{u}^\sigma(t)} \Delta t < \infty,
\]
which is a contradiction. This completes the proof. \( \square \)

References


