TAMKANG JOURNAL OF MATHEMATICS
Volume 44, Number 3, 227-232, Autumn 2013
doi:10.5556/j.tkjm.44.2013.1189
Available online at http://journals.math.tku.edu.tw/

OSCILLATION CRITERION FOR TWO-DIMENSIONAL DYNAMIC SYSTEMS ON TIME SCALES

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Abstract. The purpose of this paper is to prove oscillation criterion for dynamic system

$$u^{\Delta} = pv, \qquad v^{\Delta} = -qu^{\sigma},$$

where $p > 0$ and q are rd-continuous functions on a time scale such that $\sup \mathbb{T} = \infty$ without explicit sign assumptions on q and also without restrictive conditions on the time scale $\mathbb{T}$.

1. Introduction

We consider the linear dynamic system

$$u^{\Delta} = pv, \qquad v^{\Delta} = -qu^{\sigma}, \tag{1.1}$$

where $p > 0$ and q are rd-continuous functions on a time scale such that $\sup \mathbb{T} = \infty$. A solution $(u(t), v(t))$ of system (1.1) is called oscillatory if both $u(t)$ and $v(t)$ are oscillatory functions, and otherwise it will be called nonoscillatory. System (1.1) is called oscillatory if its solutions are oscillatory.

In [6, 1], the following oscillation theorem is obtained.

Theorem 1.1. *Assume that $q(t) \geq 0$ is rd-continuous function and*

$$\int_{t_0}^{\infty} p(t)\Delta t = \infty \qquad \textit{and} \qquad \lim_{t\to\infty} \frac{\mu(t)p(t)}{\overline{p}(t)} = 0. \tag{1.2}$$

If there exists $\lambda \in (0,1)$ such that

$$\int_{t_0}^{\infty} \overline{p}^{\lambda}(t)q(t)\Delta t = \infty,$$

where

$$\overline{p}(t) := \int_{t_0}^{t} p(s)\Delta s, \tag{1.3}$$

then system (1.1) is oscillatory.

2010 *Mathematics Subject Classification.* 34K11, 39A10, 39A99.
Key words and phrases. Oscillation, dynamic system, time scales.

Recently, Baoguo [5] proved above theorem when $q(t)$ is allowed to take on negative values and established the following theorem.

Theorem 1.2. *Assume that $\mathbb{T}$ satisfies condition (C) and* (1.2). *If there exists $\lambda \in [0,1)$ such that*

$$\int_{t_0}^{\infty} \overline{p}(\sigma(t))q(t)\Delta t = \infty, \tag{1.4}$$

where $\overline{p}(t)$ is defined by (1.3)*, then system* (1.1) *is oscillatory.*

To be precise, we say $\mathbb{T}$ satisfies condition (C), that is there is an $M > 0$ such that $\chi(t) \leq M\mu(t)$, $t \in \mathbb{T}$, where χ is the characteristic function of the set $\hat{\mathbb{T}} = \{t \in \mathbb{T} : \mu(t) > 0\}$. We note that if $\mathbb{T}$ satisfies condition (C), then the subset $\check{\mathbb{T}}$ of $\mathbb{T}$ defined by

$$\check{\mathbb{T}} = \{t \in \mathbb{T} : t > 0 \text{ is right-scattered or left-scattered}\},$$

is necessarily countable and $\hat{\mathbb{T}} \subset \check{\mathbb{T}}$. Then, we can rewrite $\check{\mathbb{T}}$ by

$$\check{\mathbb{T}} = \{t_i \in \mathbb{T} : 0 < t_1 < t_2 < \dots < t_n < \dots\},$$

and so

$$\mathbb{T} = \check{\mathbb{T}} \cup [\cup_{n \in A}(t_{n-1}, t_n)],$$

where A is the set of all integers for which the real open interval (t_{n-1}, t_n) is contained in $\mathbb{T}$. There are several time scales do not satisfy condition (C), for example
It is easy to see the time scale in the form

$$\mathbb{T} := \bigcup_{k=1}^{\infty} T_k, \text{ where } T_k = \bigcup_{n=1}^{\infty} \left\{k + \frac{n+1}{n}\right\},$$

does not satisfy condition (C).

Therefore it will be of great interest to prove Theorems 1.1 and 1.2 without explicit sign assumption on q and also without condition (C) to the same dynamic system (1.1), so our work improves and generalizes those established in [6, 5].

2. Main results

Before stating the main results, we start with the following lemmas which will play an important role in the proofs of the main results.

Lemma 2.1. *For all $\lambda \in [0,1)$ and for sufficiently large $T \geq t_0$, we have, for $t \in [T,\infty)_{\mathbb{T}}$*

$$\int_T^t p(s)\left[\overline{p}(\sigma(s))\right]^{\lambda-2} \Delta s \leq \frac{\left[\overline{p}(T)\right]^{\lambda-1}}{1-\lambda},$$

where $\overline{p}(t)$ is defined by (1.3).

Proof. By Pötzsche chain rule ([3, Theorem 1.90]), we have

$$\begin{aligned}\frac{1}{\lambda-1}\left(\left[\overline{p}(s)\right]^{\lambda-1}\right)^{\Delta} &= \int_0^1 \left[(1-h)\overline{p}(s)+h\overline{p}(\sigma(s))\right]^{\lambda-2}\, dh\, p(s)\\ &\geq \int_0^1 \left[(1-h)\overline{p}(\sigma(s))+h\overline{p}(\sigma(s))\right]^{\lambda-2}\, dh\, p(s)\\ &= \left[\overline{p}(\sigma(s))\right]^{\lambda-2} p(s).\end{aligned}$$

which yields

$$\begin{aligned}\int_T^t p(s)\left[\overline{p}(\sigma(s))\right]^{\lambda-2}\Delta s &\leq \frac{1}{\lambda-1}\int_T^t \left(\left[\overline{p}(s)\right]^{\lambda-1}\right)^{\Delta}\Delta s\\ &= \frac{1}{\lambda-1}\left\{\left[\overline{p}(t)\right]^{\lambda-1}-\left[\overline{p}(T)\right]^{\lambda-1}\right\}\\ &\leq \frac{\left[\overline{p}(T)\right]^{\lambda-1}}{1-\lambda}.\end{aligned}$$

□

Lemma 2.2. *If y and x are differentiable on $\mathbb{T}$ and $x \neq 0$ on $\mathbb{T}$, then*

$$x^{\Delta}\left(\frac{y^2}{x}\right)^{\Delta} = \left(y^{\Delta}\right)^2 - \left[\frac{\frac{x^{\Delta}y^{\sigma}}{x}}{\sqrt{1+\mu\frac{x^{\Delta}}{x}}} + \sqrt{1+\mu\frac{x^{\Delta}}{x}}\, y^{\Delta}\right]^2.$$

Proof. From the quotient rule, we get

$$x^{\Delta}\left(\frac{y^2}{x}\right)^{\Delta} = x^{\Delta}\left[\frac{x\left(y^2\right)^{\Delta}-y^2x^{\Delta}}{xx^{\sigma}}\right] = \frac{x^{\Delta}}{xx^{\sigma}}\left[x\left(y^2\right)^{\Delta}-y^2x^{\Delta}\right]$$

Since $\left(y^2\right)^{\Delta} = (yy)^{\Delta} = y^{\Delta}\left(y^{\sigma}+y\right) = y^{\Delta}\left(2y+\mu y^{\Delta}\right)$, then

$$\begin{aligned}x^{\Delta}\left(\frac{y^2}{x}\right)^{\Delta} &= \frac{x^{\Delta}}{xx^{\sigma}}\left[2xyy^{\Delta}+\mu x\left(y^{\Delta}\right)^2-x^{\Delta}y^2\right]\\ &= \frac{x^{\Delta}}{xx^{\sigma}}\left[2yy^{\Delta}\left(x^{\sigma}-\mu x^{\Delta}\right)+\mu\left(y^{\Delta}\right)^2\left(x^{\sigma}-\mu x^{\Delta}\right)-x^{\Delta}y^2\right]\\ &= \frac{x^{\Delta}}{xx^{\sigma}}\left[-x^{\Delta}\left(y^2+2\mu yy^{\Delta}+\mu^2\left(y^{\Delta}\right)^2\right)+x^{\sigma}y^{\Delta}\left(y+\mu y^{\Delta}\right)+x^{\sigma}yy^{\Delta}\right]\\ &= \frac{x^{\Delta}}{xx^{\sigma}}\left[-x^{\Delta}\left(y+\mu y^{\Delta}\right)^2+x^{\sigma}y^{\Delta}\left(y+\mu y^{\Delta}\right)+x^{\sigma}y^{\Delta}\left(y^{\sigma}-\mu y^{\Delta}\right)\right]\\ &= \frac{x^{\Delta}}{xx^{\sigma}}\left[-x^{\Delta}\left(y^{\sigma}\right)^2+2x^{\sigma}y^{\Delta}y^{\sigma}-\mu x^{\sigma}\left(y^{\Delta}\right)^2\right]\\ &= -\left(\frac{x^{\Delta}y^{\sigma}}{x}\right)^2\frac{x}{x^{\sigma}}+\frac{2x^{\Delta}}{x}y^{\Delta}y^{\sigma}-\frac{\mu x^{\Delta}}{x}\left(y^{\Delta}\right)^2\\ &= -\frac{\left(\frac{x^{\Delta}y^{\sigma}}{x}\right)^2}{1+\mu\frac{x^{\Delta}}{x}}+\frac{2x^{\Delta}}{x}y^{\Delta}y^{\sigma}-\frac{\mu x^{\Delta}}{x}\left(y^{\Delta}\right)^2\end{aligned}$$

$$= -\left[\frac{\frac{x^\Delta y^\sigma}{x}}{\sqrt{1+\mu\frac{x^\Delta}{x}}} + \sqrt{1+\mu\frac{x^\Delta}{x}}\, y^\Delta\right]^2 + \left(y^\Delta\right)^2,$$

since $1+\mu\frac{x^\Delta}{x} = \frac{x^\sigma}{x} \neq 0$. □

From above lemmas and motivated by the proof of Theorem 3.1 in [5], we can prove Theorem 1.1 without condition (C).

Theorem 2.3. *Assume that*

$$\int_{t_0}^{\infty} p(t)\,\Delta t = \infty \quad \text{and} \quad \lim_{t\to\infty}\frac{\mu(t)\,p(t)}{\overline{p}(t)} = 0. \tag{2.1}$$

If there exists $\lambda \in [0,1)$ *such that*

$$\int_{t_0}^{\infty} \overline{p}^{\lambda}(\sigma(t))q(t)\Delta t = \infty, \tag{2.2}$$

where $\overline{p}(t)$ *is defined by* (1.3)*, then system* (1.1) *is oscillatory.*

Proof. Assume that $(u(t), v(t))$ is a nonoscillatory solution of system (1.1). We claim that $u(t)$ is nonoscillatory. If not, we assume $u(t)$ is oscillatory and $v(t)$ is nonoscillatory. Without loss of generality, we let $v(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. In the view of the first equation of system (1.1), we have $u^\Delta(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Thus $u(t) > 0$ or $u(t) < 0$ for all large t, which is a contradiction. Thus $u(t)$ is nonoscillatory and without loss of generality, we let $u(t) > 0$ for $t \geq T \geq t_0$. Define, for $\lambda \in [0,1)$

$$w(t) := \frac{\overline{p}^{\lambda}(t)\,u^\Delta(t)}{p(t)\,u(t)}.$$

Then, from the product rule, we get (suppressing arguments)

$$w^\Delta = \left[\frac{\overline{p}^\lambda}{u}\right]^\sigma \left(\frac{1}{p}u^\Delta\right)^\Delta + \left[\frac{\overline{p}^\lambda}{u}\right]^\Delta \left(\frac{1}{p}u^\Delta\right) = -\left(\overline{p}^\lambda\right)^\sigma q + \frac{1}{p}\left[u^\Delta\left(\frac{\left(\overline{p}^{\lambda/2}\right)^2}{u}\right)^\Delta\right].$$

Now, by using Lemma 2.2 with replaced x by u and y by $\overline{p}^{\lambda/2}$, we get

$$\begin{aligned} w^\Delta &= -\left(\overline{p}^\lambda\right)^\sigma q + \frac{1}{p}\left[u^\Delta\left(\frac{\left(\overline{p}^{\lambda/2}\right)^2}{u}\right)^\Delta\right] \\ &= -\left(\overline{p}^\lambda\right)^\sigma q + \frac{1}{p}\left[\left(\left(\overline{p}^{\lambda/2}\right)^\Delta\right)^2 - \left[\frac{\frac{u^\Delta\left(\overline{p}^{\lambda/2}\right)^\sigma}{u}}{\sqrt{1+\mu\frac{u^\Delta}{u}}} + \sqrt{1+\mu\frac{u^\Delta}{u}}\,\overline{p}^{\lambda/2\Delta}\right]^2\right] \end{aligned}$$

$$\le -\left(\overline{p}^{\lambda}\right)^{\sigma} q+\frac{\left(\left(\overline{p}^{\lambda/2}\right)^{\Delta}\right)^{2}}{p}.$$

Therefore,

$$\int_{T}^{t} w^{\Delta}(s)\;\Delta s \le -\int_{T}^{t} \overline{p}^{\lambda}(\sigma(s))\, q(s)\;\Delta s+\int_{T}^{t} \frac{\left[\left(\overline{p}^{\lambda/2}(s)\right)^{\Delta}\right]^{2}}{p(s)}\;\Delta s. \tag{2.3}$$

Using the Pötzsche chain rule ([3, Theorem 1.90]), we get

$$\left(\overline{p}^{\lambda/2}(t)\right)^{\Delta}=\frac{\lambda}{2}\int_{0}^{1}\left[(1-h)\overline{p}(t)+h\overline{p}(\sigma(t))\right]^{\lambda/2-1}\,dh\,p(t). \tag{2.4}$$

Since $\lim_{t\to\infty}\frac{\mu(t)p(t)}{\overline{p}(t)}=0$, it implies that

$$\lim_{t\to\infty}\frac{\overline{p}(\sigma(t))}{\overline{p}(t)}=\lim_{t\to\infty}\frac{\int_{t_0}^{t}p(s)\Delta s+\int_{t}^{\sigma(t)}p(s)\Delta s}{\int_{t_0}^{t}p(s)\Delta s}=1+\lim_{t\to\infty}\frac{\mu(t)p(t)}{\overline{p}(t)}=1,$$

so for given $0<\epsilon<1$, there exists T_1 sufficiently large such that

$$\overline{p}(t)\ge(1-\epsilon)\,\overline{p}(\sigma(t)),\qquad \text{for } t\ge T_1.$$

Then, from (2.4), we have

$$\begin{aligned}\left(\overline{p}^{\lambda/2}(t)\right)^{\Delta} &\le \frac{\lambda}{2}p(t)\left[\overline{p}(\sigma(t))\right]^{\lambda/2-1}\int_{0}^{1}\left[1-(1-h)\epsilon\right]^{\lambda/2-1}\,dh\\ &= p(t)\left[\overline{p}(\sigma(t))\right]^{\lambda/2-1}\frac{1}{\epsilon}\left[1-(1-\epsilon)^{\lambda/2}\right]\\ &= Mp(t)\left[\overline{p}(\sigma(t))\right]^{\lambda/2-1},\end{aligned} \tag{2.5}$$

where $M:=\frac{1}{\epsilon}\left[1-(1-\epsilon)^{\lambda/2}\right]$. From (2.3) and (2.5), we get

$$\begin{aligned}w(t)-w(T) &\le -\int_{T}^{t}\overline{p}^{\lambda}(\sigma(s))\,q(s)\;\Delta s+\int_{T}^{t}\frac{\left[\left(\overline{p}^{\lambda/2}(s)\right)^{\Delta}\right]^{2}}{p(s)}\;\Delta s\\ &\le -\int_{T}^{t}\overline{p}^{\lambda}(\sigma(s))\,q(s)\;\Delta s+M^{2}\int_{T}^{t}p(s)\left[\overline{p}(\sigma(s))\right]^{\lambda-2}\;\Delta s.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{\lambda-1}\left(\left[\overline{p}(s)\right]^{\lambda-1}\right)^{\Delta} &= \int_{0}^{1}\left[(1-h)\overline{p}(s)+h\overline{p}(\sigma(s))\right]^{\lambda-2}\,dh\,p(s)\\ &\ge \int_{0}^{1}\left[(1-h)\overline{p}(\sigma(s))+h\overline{p}(\sigma(s))\right]^{\lambda-2}\,dh\,p(s)\\ &= \left[\overline{p}(s)\right]^{\lambda-2}p(s),\end{aligned}$$

which yields

$$w(t)-w(T) \leq -\int_T^t \overline{p}^{\lambda}(\sigma(s))\, q(s)\, \Delta s + M^2\left(\frac{[\overline{p}(t)]^{\lambda-1}}{\lambda-1}-\frac{[\overline{p}(T)]^{\lambda-1}}{\lambda-1}\right)$$
$$\leq -\int_T^t \overline{p}^{\lambda}(\sigma(s))\, q(s)\, \Delta s + M^2\left(\frac{[\overline{p}(T)]^{\lambda-1}}{1-\lambda}\right).$$

In view of condition (2.2), it follows from the last inequality that there exists a sufficiently large $T_2 \geq T_1$ such that

$$u^{\Delta}(t) < 0, \quad \text{for } t \in [T_2,\infty)_{\mathbb{T}}. \tag{2.6}$$

By using [3, Theorem 4.61], there is a positive solution $\tilde{u}$, called dominant solution such that

$$\int_{T_2}^{\infty} \frac{p(t)}{\tilde{u}(t)\, \tilde{u}^{\sigma}(t)} \Delta t < \infty,$$

and then, from (2.6), we get

$$\tilde{u}^{\Delta}(t) < 0, \quad \text{for } t \in [T_2,\infty)_{\mathbb{T}}.$$

This implies

$$\int_{T_2}^{\infty} p(t)\Delta t = \tilde{u}(T_2)\, \tilde{u}^{\sigma}(T_2) \int_{T_2}^{\infty} \frac{p(t)}{\tilde{u}(T_2)\, \tilde{u}^{\sigma}(T_2)} \Delta t$$
$$\leq \tilde{u}(T_2)\, \tilde{u}^{\sigma}(T_2) \int_{T_2}^{\infty} \frac{p(t)}{\tilde{u}(t)\, \tilde{u}^{\sigma}(t)} \Delta t < \infty,$$

which is a contradiction. This completes the proof. □

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