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DIFFERENTIAL EQUATIONS AND FOLDING OF n-MANIFOLDS

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Abstract. In this paper we will describe some topological and geometric characters of n-manifold by using the properties of differential equations. The folding and unfolding of n-manifold into itself will be deduced from viewpoint of the differential equations.

Definition 1. An n-dimensional manifold is a Housdorff space such that each point has an open neighbourhood homeomorphic to the open n-dimensional disc $\cup^n(= \{x \in \mathcal{R}^n : |x| < 1\})$ [10, 14].

Definition 2. The set is compact if it closed and bounded [10].

Definition 3. For Riemannian manifolds M and N (not necessarily of the same dimension), a map $f : M \to N$ is said to be a "*topological folding*" of M into N if, for each piecewise geodesic path $\gamma : I \to M$ $(I = [0, 1] \subseteq R)$, the induced path $f \circ \gamma : I \to N$ is piecewise geodesic. If, in addition, $f : M \to N$ preserves lengths of paths, we call f an "*isometric folding*" of M into N. Thus an isometric folding is necessarily a topological folding. Some types of folding of manifolds discussed in [3, 6] and some applications of foldings introduced in [8].

Definition 4. Let M and N be two Riemannian manifolds of the same dimensions, a map $g : M \to N$ is said to be an "*unfolding*" of M into N if, for every piecwise geodesic path $\gamma : I \to M$ $(I = [0, 1] \subseteq R)$, the induced path $\gamma' = g \circ \gamma : I \to N$ is piecewise geodesic but with length greater than that for γ.

i.e. $$\forall x, y \in M \Rightarrow d(x, y) \le d(g(x), g(y))[2, 4, 5].$$

Theorem 1. *If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A are real and distinct, then any set of corresponding eigenvectors $\{V_1, V_2, \ldots, V_n\}$ forms a basis for $\mathcal{R}^n$, the matrix*

$$p = [V_1 V_2 \cdots V_n] \text{ is invertible, } p^{-1}Ap = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]\ [12, 13].$$

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Theorem 2. *If the $2n \times 2n$ real matrix A has $2n$ distinct complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding complex eigenvectors $w_j = u_j + iV_j$ and $\bar{w}_j = u_j - iV_j$, $j = 1, \ldots, n$, then $\{u_1 V_1, \ldots, u_n V_n\}$ is a basis for $\mathcal{R}^{2n}$ the matrix $p = [u_1 V_1, \ldots, u_n, V_n]$ is invertible and*

$$p^{-1}Ap = \operatorname{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$$

A real $2n \times 2n$ matrix with 2×2 blocks along the diagonal.

The main results.

In this article we restrict on surfaces which represent 2-manifold and some examples of n-manifolds.

Theorem 3. *The folding of any n-manifold represented by $D(V) = AV$ is a restriction on the elements of $n \times n$ matrix $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ or on $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real (complex) and distinct eignevalues of $n \times n$ matrix A, V is vector in R^n and $|\lambda_i| < 1$.*

Proof. Let

$$D(V) = AV \tag{1}$$

i.e. $\dot{V} = AV$

let $V = (x_1, x_2, \ldots, x_n)$, then $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

the equation (1) can be written in the following form

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2}$$

from Theorems 1 and 2 (according to $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real or complex) we have the following system equivalent to the system (2)

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and any $\boldsymbol{n} \times \boldsymbol{n}$ matrix equivalent to the matrix

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix}$$

If $|b_1| \le 1$, $|b_2| \le 1, \dots, |b_n| \le 1$ then all vectors $(y_1, y_2, \dots, y_n)$ folded.

Also
$$\ddot{Y} = B\dot{Y} = B^2 Y = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} b_1^2 & 0 & \cdots & 0 \\ 0 & b_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

if $|b_1|, |b_2|, \dots, |b_n| < 1$ then $\ddot{Y}$ represent the folding of Y

also
$$\dddot{Y} = B^3 Y = \begin{bmatrix} b_1^3 & 0 & \cdots & 0 \\ 0 & b_2^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$Y^{(n)} = B^n Y = \begin{bmatrix} b_1^n & 0 & \cdots & 0 \\ 0 & b_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

this is a sequence of foldings f_n of Y which also is foldings of X.

Theorem 4. *Let $X_1, X_2, \dots, X_n$ are functions of $w_1, w_2, \dots, w_n$, i.e. $X_i = X_i(w_1, w_2, \dots, w_n)$, $i \in \{1, \dots, n\}$ and $X_i = W_{1i}(w_1)W_{2i}(w_2)\cdots W_{ni}(w_n)$ then the folding of any n-manifold represented by $f(X) = D_{wj}X = AX$, where $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ and $j \in \{1, \dots, n\}$, is a restraction on the elements of $n \times n$ matrix $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ or on the distinct and real (complex) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the $n \times n$ matrix A.*

Proof. Let $X_1, X_2, \dots, X_n$ are functions of $w_1, w_2, \dots, w_n$, i.e. $X_i = X_i(w_1, w_2, \dots, w_n)$, $i \in \{1, \dots, n\}$ and $X_i = W_{1i}(w_1)W_{2i}(w_2)\cdots W_{ni}(w_n)$, $f : M \to M$ be a folding where $f \equiv D_{wj}$. Let we have the system

$$D_{wj}X = AX \tag{1}$$

where $$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

the system (1) can be written in the form

$$D_{wj} \begin{bmatrix} W_{11}(w_1) & W_{21}(w_2) & \cdots & W_{n1}(w_n) \\ W_{12}(w_1) & W_{22}(w_2) & \cdots & W_{n2}(w_n) \\ & \vdots & & \\ W_{1n}(w_1) & W_{2n}(w_2) & \cdots & W_{nn}(w_n) \end{bmatrix}$$
$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} W_{11}(w_1) & W_{21}(w_2) & \cdots & W_{n1}(w_n) \\ W_{12}(w_1) & W_{22}(w_2) & \cdots & W_{n2}(w_n) \\ & \vdots & & \\ W_{1n}(w_1) & W_{2n}(w_2) & \cdots & W_{nn}(w_n) \end{bmatrix}$$

the above system can be reduced to the form

$$\begin{bmatrix} \dot{W}_{j1} \\ \dot{W}_{j2} \\ \vdots \\ \dot{W}_{jn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} W_{j1} \\ W_{j2} \\ \vdots \\ W_{jn} \end{bmatrix} \tag{2}$$

from theorem (3) we find that the folding of any n-manifold into itself represented by the system (2) is a restriction on the elements of the matrix $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ or on the distinct and real (complex) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the $n \times n$ matrix A. Then also the folding of any n-manifold represented by $D_{wj}X = AX$ is a restriction on the elements of the matrix $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ or on the distinct and real (complex) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the $n \times n$ matrix A.

Corollary. *The limit of the foldings of the conditional folding in Theorem* 3 *is a manifold of dimension k, $k < n$.*

Proof. Let $D^m Y = f_m(Y)$, $m = 1, 2, \ldots$, we have a sequence $f_1, f_2, \ldots, f_m$.

$$\text{Then} \qquad D^m Y = \begin{bmatrix} b_1^m & 0 & \cdots & 0 \\ 0 & b_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1^m y_1 \\ b_2^m y_2 \\ \vdots \\ b_n^m y_n \end{bmatrix} = (b_1^m y_1, b_2^m y_2, \ldots, b_n^m y_n).$$

Such that $|b_i^m| << 1$, $\forall i \in \{1, \ldots, n\}$ also $|b_k^m| << 1$ where $k \in \{1, \ldots, n\}$. Let $b_j \to 0$ as $m \to \infty$ then $\lim_{m\to\infty} f_m(Y) = (\xi_1 y_1, \xi_2 y_2, \ldots, 0, \ldots, \xi_n y_n) \in R^{n-1}$, where $b_j <<< b_i$, $\forall i \in \{1, \ldots, n\}$ and $\lim_{m\to\infty} b_j^m \to 0$ faster than the other b_i, $i \neq j$. Then the manifold must be of dimension k such that $\max k = n - 1$. See Figure 1.

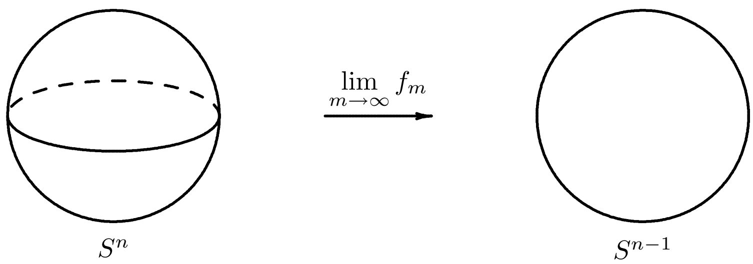


Figure 1.

Corollary. *The end of the limit of foldings in Theorem* 3 *is a* 0*-manifold.*

Proof.

$$\text{Let} \qquad f_m(Y) = D^m Y = \begin{bmatrix} b_1^m & 0 & \cdots & 0 \\ 0 & b_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n^m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1^m y_1 \\ b_2^m y_2 \\ \vdots \\ b_n^m y_n \end{bmatrix} = (b_1^m y_1, b_2^m y_2, \ldots, b_n^m y_n)$$

$$\text{Since} \quad \lim_{m\to\infty} f_m(Y) = (\xi_1 y_1, \xi_2 y_2, \ldots, 0, \xi_{k+1} y_{k+1}, \ldots, \xi_n y_n)$$

$$f_{m1} = \lim_{m\to\infty} f_m(Y) = (\xi_1 y_1, \ldots, 0, \xi_{k+1} y_{k+1}, \ldots, \xi_n y_n),$$

$$f_{m2} = (\xi_1 y_1, \ldots, 0, 0, \xi_{k+1} y_{k+1}, \ldots, \xi_n y_n)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\lim_{k\to\infty} f_{mk}(y) = (0, 0, \ldots, 0) \quad \text{which is 0-manifold (See Figure 2).}$$

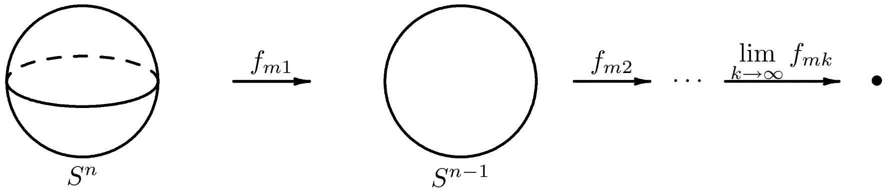


Figure 2.

From the above theorem, the two types of differentiations X_θ, X_ϕ, where $X = (x(\theta,\phi), y(\theta,\phi), z(\theta,\phi))$, the first is a limit of foldings, the second is a folding. The first type induces a limit of foldings of the tangent space, the second type induces a folding of the tangent space.

Example 1. The parametric equations of the unit sphere S^2 given by

$$x = \cos\theta\sin\phi$$
$$y = \sin\theta\sin\phi$$
$$z = \cos\phi$$

where $0 < \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$ by differentiating the above system with respect to θ and ϕ we have the boundary-value partial differential equations of the unit sphere S^2.

$$x_\theta = -y \quad 0 < \theta \leq 2\pi,\ 0 \leq \phi \leq \pi$$
$$y_\theta = x$$
$$z_\theta = 0$$
$$x_\phi = \cos\theta\sin\phi$$
$$y_\phi = \sin\theta\sin\phi$$
$$z_\phi = -\sin\phi$$

the boundary condations are

$$x(\theta,0) = x(\theta,\pi) = 0 \qquad 0 < \theta \leq 2\pi$$
$$y(\theta,0) = y(\theta,\pi) = 0$$
$$z(\theta,0) = 1, \qquad z(\theta,\pi) = -1$$

if we take the part

$$X_\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X$$

we find the eigenvalues of the matrix $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are $\lambda_{1,2} = \pm i$ and $\lambda_3 = 0$ if we take $|\lambda_1| < 1$, $|\lambda_2| < 1$ we have a limit folding of the unit sphere S^2 and its tangent space (see Figure 3).

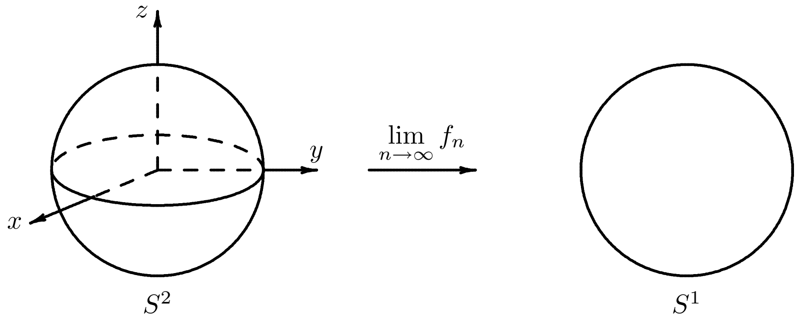


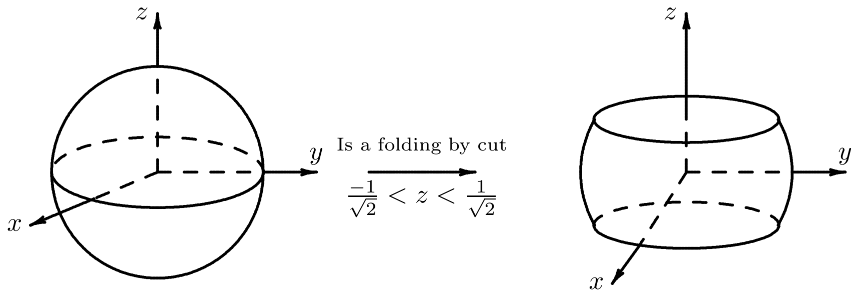
Figure 3.

A restriction of the boundary conditions in the boundary—value partial differential equations and the singularity of the folding at this restriction also making folding. The following example show this idea,

Example 2. From Example 1 by doing restriction of the boundary conditions we will have many cases of foldings of the unit sphere S^2. Now we will show those cases,

Case 1. If we take the boundary conditions in the form

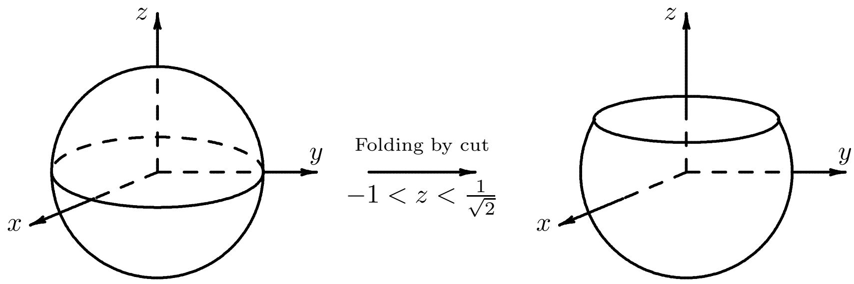
$$\begin{aligned} x\left(\theta, \frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\cos\theta, & x\left(\theta, \frac{3\pi}{4}\right) &= \frac{1}{\sqrt{2}}\cos\theta, \\ y\left(\theta, \frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\sin\theta, & y\left(\theta, \frac{3\pi}{4}\right) &= \frac{1}{\sqrt{2}}\sin\theta, \\ z\left(\theta, \frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}, & z\left(\theta, \frac{3\pi}{4}\right) &= \frac{-1}{\sqrt{2}}, \\ 0 < \theta &\le 2\pi, & \frac{\pi}{4} \le \phi &\le \frac{3\pi}{4}, \end{aligned}$$



the result sphere is 2-manifold with boundary.

Case 2. If we change the boundary conditions into

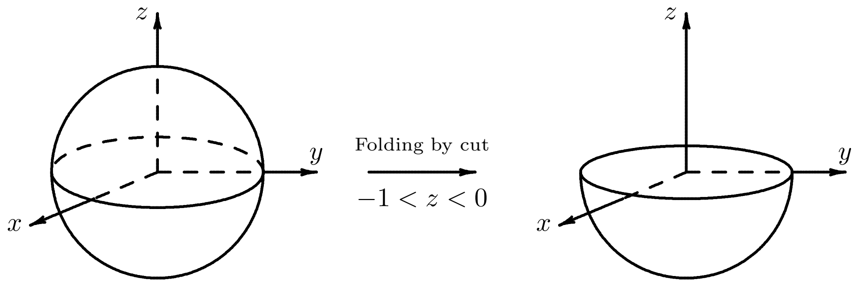
$$
\begin{aligned}
x\left(\theta,\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\cos\theta, & x(\theta,\pi) &= 0,\\
y\left(\theta,\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}\sin\theta, & y(\theta,\pi) &= 0,\\
z\left(\theta,\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}}, & z(\theta,\pi) &= -1,\\
0 < \theta &\le 2\pi, & \frac{\pi}{4} &\le \phi \le \pi,
\end{aligned}
$$



the graph after folding is 2-manifold

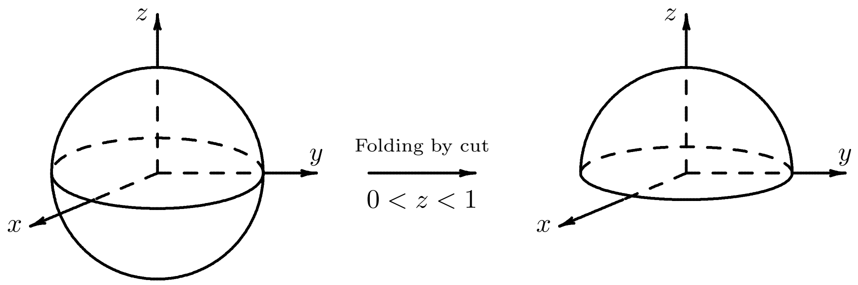
Case 3. If the boundary conditions are

$$
\begin{aligned}
x\left(\theta,\frac{\pi}{2}\right) &= \cos\theta, & x(\theta,\pi) &= 0,\\
y\left(\theta,\frac{\pi}{2}\right) &= \sin\theta, & y(\theta,\pi) &= 0,\\
z\left(\theta,\frac{\pi}{2}\right) &= 0, & z(\theta,\pi) &= -1,\\
0 < \theta &\le 2\pi, & \frac{\pi}{2} &\le \phi \le \pi,
\end{aligned}
$$



Case 4. Consider the boundary conditions on the form

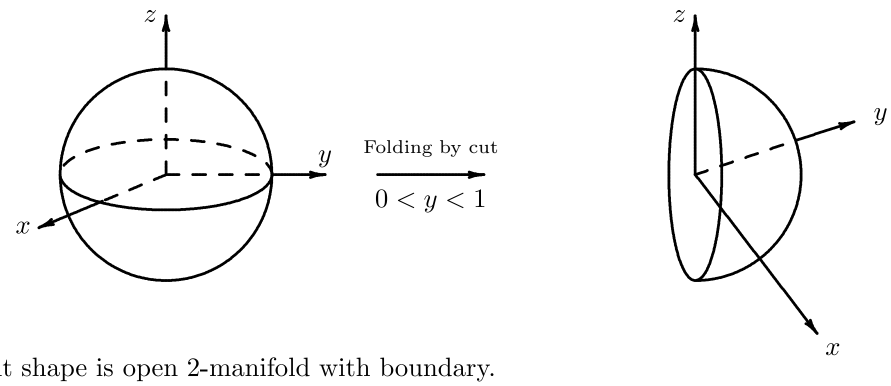
$$\begin{aligned} x(\theta,0)&=0, & x\left(\theta,\frac{\pi}{2}\right)&=\cos\theta,\\ y(\theta,0)&=0, & y\left(\theta,\frac{\pi}{2}\right)&=\sin\theta,\\ z(\theta,0)&=1, & z\left(\theta,\frac{\pi}{2}\right)&=0,\\ 0<\theta&\le 2\pi, & 0\le\phi&\le\frac{\pi}{2}, \end{aligned}$$



the result graph is open 2-manifold without boundary.

Case 5. If we take the boundary conditions in the form

$$\begin{aligned} x(0,\phi)&=\sin\phi, & x(\pi,\phi)&=-\sin\phi,\\ y(0,\phi)&=0, & y(\pi,\phi)&=0,\\ z(0,\phi)&=\cos\ \phi, & z(\pi,\phi)&=\cos\phi,\\ 0<\theta&<\pi, & 0\le\phi&\le\pi, \end{aligned}$$



the result shape is open 2-manifold with boundary.

Note 1. From the boundary conditions of the Case 1−5 we find that the limit of the foldings of the unit sphere is a circle (see Figure 4).

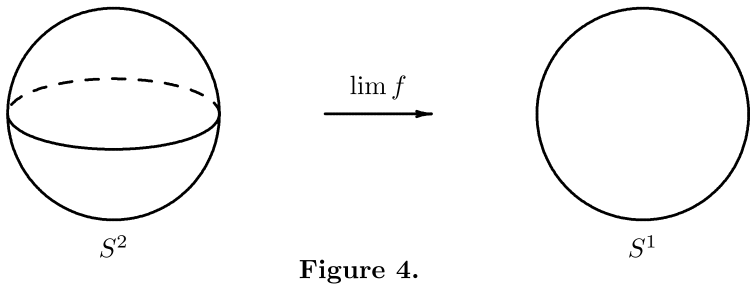


Figure 4.

Case 6. If we take $-1 < x < 1$, $0 < y < 1$, $\frac{-1}{\sqrt{2}} < z < \frac{1}{\sqrt{2}}$ where $0 < \theta < \pi$, $\frac{\pi}{4} < \phi < \frac{3\pi}{4}$ we find that the shape after folding is 2-manifold with boundary (see Figure 5).

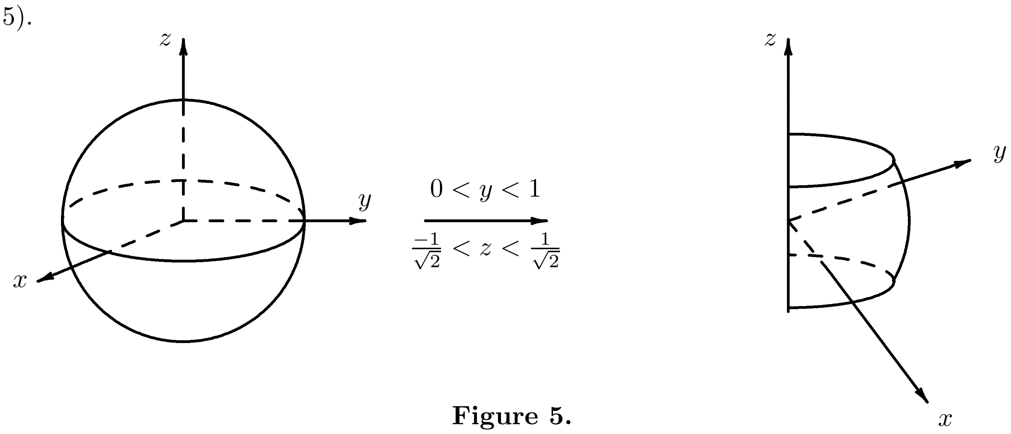


Figure 5.

The above cases are not only the cases of foldings of S^2 but there are many cases also are foldings of S^2.

Note 2. The conditions of Case 6 represent the end of the limit foldings of the unit sphere S^2 which is a tow points (see Figure 6).

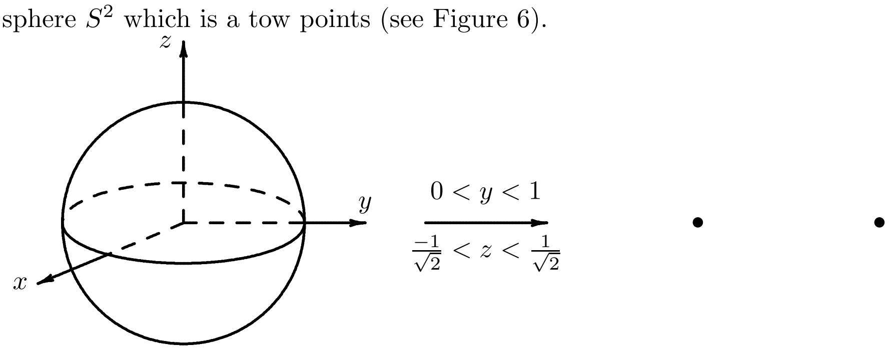


Figure 6.

Note 3. We note from the Case 1−6 that a restriction on one of the parameter θ or Φ is folding the sphere S^2 into a circle but a restriction on the two parameters θ and Φ is folding the sphere S^2 into two points.

Example 3. Consider the parametric equations of tours T^1 given by

$$\begin{aligned} x_1 &= (b + a\cos u)\cos v, \qquad 0 < v < 2\pi,\ 0 < u < 2\pi, \\ x_2 &= (b + a\cos u)\sin v, \\ x_3 &= a\sin u, \end{aligned}$$

the boundary-value partial differential equations of the tours T^1 are

$$\begin{aligned} D_V x_1 &= -x_2 \\ D_V x_2 &= x_1 \\ D_V x_3 &= 0 \\ D_u x_1 &= a\sin u\cos(v+\pi) \\ D_u x_1 &= a\sin u\sin(v+\pi) \\ D_u x_3 &= a\cos u \\ x_1(v,0) &= (b+a)\cos v \\ x_2(v,0) &= (b+a)\sin v \\ x_3(v,0) &= 0 \\ 0 < v < 2\pi&, \qquad 0 < u < 2\pi \end{aligned}$$

if we take the part

$$X_v = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X$$

the eigenvalues of the matrix $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are $\lambda_{1,2} = \pm i$, $\lambda_3 = 0$ which represent a limit of $\lim f_n\{T^1 - S^1\}$ (see Figure 7). If we take $|\lambda_1| < 1$, $|\lambda_2| < 1$ the tours T^1 folding into itself and also if $|\lambda_1| > 1$, $|\lambda_2| > 1$ the tours T^1 unfolding into itself.

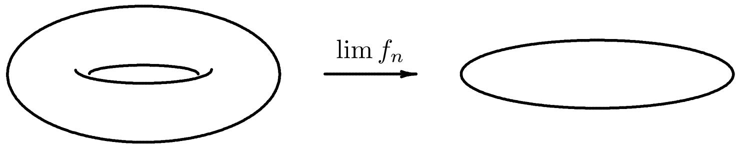


Figure 7.

If we take the part

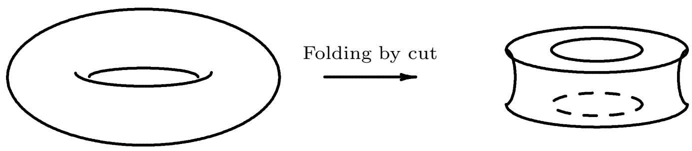
$$D_u x_1 = a \sin u \cos(v + \pi)$$
$$D_u x_1 = a \sin u \sin(v + \pi)$$
$$D_u x_3 = a \cos u$$

we find it a type of folding.

Example 4. In this example we will show some of folding T^1 which corresponding to the cases in example two.

Case 1. If we take the boundary conditions in the form

$$x_1\left(v, \frac{\pi}{2}\right) = b \cos v, \qquad x_1\left(v, \frac{3\pi}{2}\right) = b \cos v,$$
$$x_2\left(v, \frac{\pi}{2}\right) = b \sin v, \qquad x_2\left(v, \frac{3\pi}{2}\right) = b \cos v,$$
$$x_3\left(v, \frac{\pi}{2}\right) = a, \qquad x_3\left(v, \frac{3\pi}{2}\right) = -a,$$
$$\frac{\pi}{2} < u < \frac{3\pi}{2}, \qquad 0 < v < 2\pi,$$



the result graph is 2-manifold.

Case 2. If we take the boundary conditions in the form

$$x_1(v, 0) = (b + a) \cos v, \qquad x_1(v, \pi) = (b - a) \cos v,$$
$$x_2(v, 0) = (b + a) \sin v, \qquad x_2(v, \pi) = (b - a) \cos v,$$
$$x_3(v, 0) = 0, \qquad x_3(v, \pi) = 0$$
$$0 < u < \pi, \qquad 0 < v < 2\pi,$$

the graph after folding by cut is the upper half of the tours T^1 which is 2-manifold.

Case 3. This case corresponding to the Case 4 in Example 2 the boundary condition are

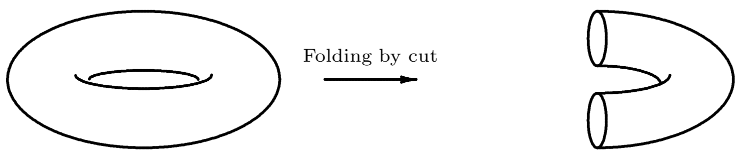
$$x_1(v, \pi) = (b - a) \cos v, \qquad x_1(v, 2\pi) = (b + a) \cos v,$$

$$x_2(v,\pi) = (b-a)\sin v, \qquad x_2(v,2\pi) = (b+a)\sin v,$$
$$x_3(v,\pi) = 0, \qquad x_3(v,2\pi) = 0,$$
$$\pi < u < 2\pi, \qquad 0 < v < 2\pi$$

the graph after folding by cut is the upper half of the tours T^1 which is 2-manifold.

Case 4. This case corresponding to the Case 5 in Example 2 the boundary condition are

$$x_1(0,u) = b + a\cos u, \qquad x_1(2\pi,v) = -(b + a\cos u),$$
$$x_2(0,u) = 0, \qquad x_2(\pi,u) = 0,$$
$$x_3(0,u) = a\sin u, \qquad x_3(\pi,u) = a\sin u,$$
$$0 < u < 2\pi, \qquad 0 < v < \pi,$$



the graph after folding is 2-manifold with boundary.

Note 4. From the boundary conditions of the Case 1−4 above we conclude that the limit folding of the tours T^1 is a circle (see Figure 7).

Case 5. If we make a restriction on u and v as $\frac{\pi}{2} < u < \frac{3\pi}{2}$, $0 < v < \pi$ we have a folding by cut (see the following Figure 8).

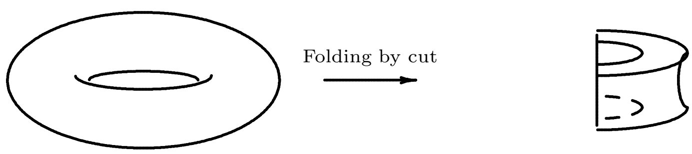


Figure 8.

the result graph is 2-manifold.

Note 5. There is a homeomorphism between folding of the unit sphere S^2 and folding of the tours T^1.

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