TAMKANG JOURNAL OF MATHEMATICS
Volume 36, Number 2, 87-92, Summer 2005

A UNIFIED INVERSE LAPLACE TRANSFORM FORMULA INVOLVING THE PRODUCT OF A GENERAL CLASS OF POLYNOMIALS AND THE FOX H-FUNCTION

R. C. SONI AND DEEPIKA SINGH

Abstract. In the present paper we obtain the inverse Laplace transform of the product of a general class of polynomials and the Fox H-function. The polynomials and the functions involved in our main formula as well as their arguments are quite general in nature. Therefore, the inverse Laplace transform of the product of a large variety of polynomials and numerous simple special functions can be obtained as simple special cases of our main result. The results obtained by Gupta and Soni [2] and Srivastava [5] follow as special cases of our main result.

1. Introduction

The Laplace transform of the function $f(t)$ is defined in the following usual manner

$$F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt. \tag{1.1}$$

The function $f(t)$ is called the inverse Laplace transform of $F(s)$ and will be denoted by $L^{-1}\{F(s)\}$ in the paper.

Also, $S_n^m[x]$ occurring in the sequal denotes the general class of polynomials introduced by Srivastava [5, p.1, Eq.(1)]

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k} (n, k \geq 0)$ are arbitrary constants, real or complex. On suitably sepcializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [8, pp.158-161].

Received June 18, 2002; revised June 3, 2004.
2000 *Mathematics Subject Classification*. 44A10, 33C45, 33C60, 33C70.
Key words and phrases. The Laplace transform, a general class of polynomials, the Fox H-function, the multivariable H-function.

The function $H[z]$ occurring in the paper stands for the Fox H-function [1] defined and represented in the following manner [6, p.10, Eq.(2.1.1)]

$$H[z] = H^{M,N}_{P,Q}\left[z \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{I,Q}\end{matrix}\right] = \frac{1}{2\pi\omega}\int_L \phi(\xi) z^{\xi} d\xi \tag{1.3}$$

where $\omega = \sqrt{-1}$,

$$\phi(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=M+1}^{Q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}. \tag{1.4}$$

The H-function of r complex variables $z_1,\ldots,z_r$ was introduced by Srivastava and Panda [7]. We shall define and represent it in the following form [6, p.251, Eq.(C.1)]

$$H[z_1,\ldots,z_r] = H^{0,N:M',N';\cdots;M^{(r)},N^{(r)}}_{P,Q:P',Q';\cdots;P^{(r)},Q^{(r)}}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\,\middle|\, \begin{matrix}(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j)_{1,P}:(c'_j,\gamma'_j)_{1,P'};\cdots;(c^{(r)}_j,\gamma^{(r)}_j)_{1,P^{(r)}}\\ (b_j;\beta'_j,\ldots,\beta^{(r)}_j)_{1,Q}:(d'_j,\delta'_j)_{1,Q'};\cdots;(d^{(r)}_j,\delta^{(r)}_j)_{1,Q^{(r)}}\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\cdots\phi_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}d\xi_1\cdots d\xi_r \tag{1.5}$$

where

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M^{(i)}}\Gamma(d^{(i)}_j-\delta^{(i)}_j\xi_i)\prod_{j=1}^{N^{(i)}}\Gamma(1-c^{(i)}_j+\gamma^{(i)}_j\xi_i)}{\prod_{j=M^{(i)}+1}^{Q^{(i)}}\Gamma(1-d^{(i)}_j+\delta^{(i)}_j\xi_i)\prod_{j=N^{(i)}+1}^{P^{(i)}}\Gamma(c^{(i)}_j-\gamma^{(i)}_j\xi_i)} \quad \forall i\in\{1,\ldots,r\} \tag{1.6}$$

$$\psi(\xi_1,\ldots,\xi_r) = \frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha^{(i)}_j\xi_i)}{\prod_{j=N+1}^{P}\Gamma(a_j-\sum_{i=1}^{r}\alpha^{(i)}_j\xi_i)\prod_{j=1}^{Q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta^{(i)}_j\xi_i)}. \tag{1.7}$$

The nature of contour L in (1.3) and contours $L_1,\ldots,L_r$ in (1.5), the various special cases and other details of the above functions can be found in the book referred to above. It may be remarked here that all the Greek letters occurring in the left-hand side of (1.3) and (1.5) are assumed to be positive real numbers for standardization purposes;

the definitions of these functions will, however, be meaningful even if some of these quantities are zero. Again, it is assumed that the Fox H-function and the H-function of serveral variables occurring in the paper always satisfy their appropriate conditions of convergence [6, pp.12-13, Eqs.(2.2.1), (2.2.11); pp.252-253, Eqs.(C.4-C.6) respectively].

2. Main Result

$$L^{-1}\left\{ s^{-\rho}\prod_{i=1}^{\tau}(s^{l_i}+\alpha_i)^{-\sigma_i} S_{n_1}^{m_1}\left[e_1 s^{-\lambda_1}\prod_{i=1}^{\tau}(s^{l_i}+\alpha_i)^{-\eta_i'}\right] S_{n_2}^{m_2}\left[e_2 s^{-\lambda_2}\prod_{i=1}^{\tau}(s^{l_i}+\alpha_i)^{-\eta_i''}\right] \right.$$
$$\left. \times H_{P,Q}^{M,N}\left[zs^{-u}\prod_{i=1}^{\tau}(s^{l_i}+\alpha_i)^{-v_i} \,\middle|\, \begin{matrix}(a_j,\alpha_j)_{1,P}\\ (b_j,\beta_j)_{1,Q}\end{matrix}\right]\right\}$$

$$= t^{\rho+l_1\sigma_1+\cdots+l_\tau\sigma_\tau-1}\sum_{k_1=0}^{[n_1/m_1]}\sum_{k_2=0}^{[n_2/m_2]}\frac{(-n_1)_{m_1k_1}(-n_2)_{m_2k_2}}{k_1!k_2!}A'_{n_1,k_1}A''_{n_2,k_2}(e_1t^{\lambda_1+\eta_1'l_1+\cdots+\eta_\tau'l_\tau})^{k_1}$$
$$(e_2t^{\lambda_2+\eta_1''l_1+\cdots+\eta_\tau''l_\tau})^{k_2} H_{\tau,1:P,Q+\tau;0,1;\cdots;0,1}^{0,\tau:M,N;1,0;\cdots;1,0}\left[\begin{matrix} zt^{u+v_1l_1+\cdots+v_\tau l_\tau}\\ \alpha_1 t^{l_1}\\ \vdots \\ \alpha_\tau t^{l_\tau}\end{matrix}\right|$$

$$(1-\sigma_1-\eta_1'k_1-\eta_1''k_2; v_1, 1, \tfrac{0,\ldots,0}{\tau-1}),\ldots,(1-\sigma_\tau-\eta_\tau'k_1-\eta_\tau''k_2; v_\tau, \tfrac{0,\ldots,0}{\tau-1}, 1):$$
$$(1-\rho-l_1\sigma_1-\cdots-l_\tau\sigma_\tau-(\lambda_1+\eta_1'l_1+\cdots+\eta_\tau'l_\tau)k_1-(\lambda_2+\eta_1''l_1+\cdots+\eta_\tau''l_\tau)k_2;$$
$$u+v_1l_1+\cdots+v_\tau l_\tau, l_1,\ldots,l_\tau):$$

$$\left.\begin{matrix}(a_j,\alpha_j)_{1,P} & ;\underline{\quad};\cdots;\underline{\quad}\\ (b_j,\beta_j)_{1,Q},(1-\sigma_1-\eta_1'k_1-\eta_1''k_2;v_1),\ldots,(1-\sigma_\tau-\eta_\tau'k_1-\eta_\tau''k_2;v_\tau) & ;\frac{(0,1);\cdots;(0,1)}{\tau}\end{matrix}\right] \tag{2.1}$$

where the function occurring on the left-hand side of (2.1) is the Fox H-function and that on its right-hand side is the H-function of $\tau+1$ variables and the following conditions are astisfied.

The quantities $\lambda_1,\eta_1',\ldots,\eta_\tau',\lambda_2,\eta_1'',\ldots,\eta_\tau'',u,v_1,\ldots,v_\tau$ are all positive, $\mathrm{Re}(s)>0$,

$$\mathrm{Re}(\rho+l_1\sigma_1+\cdots+l_\tau\sigma_\tau)+\min_{1\le j\le M}\left[\mathrm{Re}(u+v_1l_1+\cdots+v_\tau l_\tau)\left(\frac{b_j}{\beta_j}\right)\right]>0,$$
$$\sum_{j=1}^{P}\alpha_j-\sum_{j=1}^{Q}\beta_j-(u+v_1l_1+\cdots+v_\tau l_\tau)<0;$$
$$U=\sum_{j=1}^{N}\alpha_j-\sum_{j=N+1}^{P}\alpha_j+\sum_{j=1}^{M}\beta_j-\sum_{j=M+1}^{Q}\beta_j-[u+v_1(l_1+1)+\cdots+v_\tau(l_\tau+1)]>0,$$

$|\arg z| < \left(\frac{1}{2}\right) U\pi$ or $U = 0$ and $z > 0$;

$0 < l_i < 1, |\arg \alpha_i| < (1-l_i)\frac{\pi}{2}, \quad i = 1, \ldots, \tau$ or $l_1 = \cdots = l_\tau = 1$ and $\alpha_1 > 0, \ldots, \alpha_\tau > 0$.

It may be remarked here that some of the exponents $l_1, \ldots, l_\tau, \lambda_1, \eta_1', \ldots, \eta_\tau', \lambda_2, \eta_1'', \ldots, \eta_\tau'', u, v_1, \ldots, v_\tau$ in (2.1) can also decrease to zero provided that both sides of the resulting equation have a meaning. Also the number occurring below the line at any place on the right-hand side of (2.1) indicates the total number of zeros/pairs covered by it. Thus $\frac{0,\ldots,0}{r}$, $\frac{(0,1);\cdots;(0,1)}{r}$ would mean r zeros $/r$ pairs, and so on.

Proof. We first express the product of a general class of polynomials occurring on the left-hand side of (2.1) in the series form given by (1.2), replace the Fox H-function occurring therein by its well known Mellin-Barnes contour integral with the help of (1.3). Now we interchange the orders of summations and integration (which is permissible under the conditions stated with (2.1)), find the inverse Laplace transform of the result thus obtained by making use of the following known formula [3, p.12, Eq.(12)]

$$L^{-1}\left\{ s^{\sum_{i=1}^{\tau} l_i a_i - \lambda} \prod_{i=1}^{\tau} (s^{l_i} + \lambda_i)^{-a_i} \right\}$$

$$= \frac{t^{\lambda-1}}{\prod_{i=1}^{\tau} \Gamma(a_i)} H^{0,0:1,1;\cdots;1,1}_{0,1:1,1;\cdots;1,1} \left[\begin{matrix} \lambda_1 t^{l_1} \\ \vdots \\ \lambda_\tau t^{l_\tau} \end{matrix} \middle| \begin{matrix} \text{———} : (1-a_1, 1); \cdots; (1-a_\tau, 1) \\ (1-\lambda; l_1, \ldots, l_\tau) : (0,1); \cdots; (0,1) \end{matrix} \right] \tag{2.2}$$

where

$$\mathrm{Re}(s) > 0, \quad \mathrm{Re}(\lambda) > 0, \quad 0 < l_i < 2, \quad |\arg \lambda_i| < (2 - l_i)\frac{\pi}{2}$$

or $l_i = 2$ and $\lambda_i > 0$, $i = 1, 2, \ldots, \tau$, express the multivariable H-function thus obtained in terms of Mellin-Barnes contour integral with the help of (1.5) and reinterpret the resulting contour integral thus obtained in terms of the multivariable H-function. We arrive at the desired result (2.1) after a little simplificiation.

3. Special Cases and Applications

On account of the most general nature of our main result, several known and new results, follow as its special cases. The results given herein and many others which are not recorded here specifically can find important applications in boundary value problems occurring in certain fields of science and engineering. The inverse Laplace transform formula (2.1) established here is unified in nature and acts as a key formula. Thus the general class of polynomials involved in the formula (2.1) reduce to a large spectrum of polynomials listed by Srivastava and Singh [8, pp.158-161], and so from the formula (2.1) we can further obtain various inverse Laplace transform involving a number of simpler polynomials. Again, the Fox H-function [1] occurring in this formula can be suitably specialized to a remarkably wide variety of useful functions; thus the table listing various special cases of the H-function [4, pp.145-159] can be used to derive from this inverse Laplace transform formula a number of other inverse Laplace transform formulae involving any of these simpler special functions. Thus if we take $\tau = 2$ in (2.1), we get a known result obtained by Gupta and Soni [2, pp.2-3, Eq.(2.1)], which on further specialization gives a result obtained by Srivastava [5, pp.1-2, Eq.(2)] but expressed in a different form.

Now we give two new and interesting inverse Laplace transforms that follow as special cases of (2.1)

(i) On reducing the general class of polynomials $S_{n_1}^{m_1}$ and $S_{n_2}^{m_2}$ occurring in the left-hand side of (2.1) to the Hermite polynomials [8, p.158, Eq.(1.4)] and to the Laguerre polynomials [8, p.159, Eq.(1.8)] respectively, (2.1) gives the following interesting inverse Laplace transform after a little simplificatoin

$$L^{-1}\left\{s^{-(\rho+\frac{n_1}{2})}\prod_{i=1}^{\tau}(s^{l_1}+\alpha_i)^{-\sigma_i}H_{n_1}\left[\frac{\sqrt{s}}{2}\right]L_{(n_2)}^{(\alpha)}\left[\frac{1}{s}\right]H_{P,Q}^{M,N}\left[zs^{-u}\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right]\right\}$$

$$=\frac{t^{\rho+l_1\rho_1+\cdots+l_\tau\sigma_\tau-1}}{\Gamma(\sigma_1)\cdots\Gamma(\sigma_\tau)}\sum_{k_1=0}^{[n_1/2]}\sum_{k_2=0}^{n_2}\frac{(-n_1)_{2k_1}(-n_2)_{k_2}}{k_1!k_2!}(-1)^{k_1}\binom{n_2+\alpha}{n_2}\frac{1}{(\alpha+1)_{k_2}}t^{k_1+k_2}$$

$$H_{0,1:P,Q;1,1;\cdots;1,1}^{0,0:M,N;1,1;\cdots;1,1}\left[\begin{matrix}zt^u\\\alpha_1t^{l_1}\\\vdots\\\alpha_\tau t^{l_\tau}\end{matrix}\left|\begin{matrix}\text{———} : (a_j,\alpha_j)_{1,P};\ (1-\sigma_1,1);\cdots;(1-\sigma_\tau,1)\\(1-\rho-l_1\sigma_1-\cdots-l_\tau\sigma_\tau-k_1-k_2;u,l_1,\ldots,l_\tau):(b_j;\beta_j)_{1,Q};\ (0,1)\ ;\cdots;(0,1)\end{matrix}\right.\right] \tag{3.1}$$

provided that the conditoins easily obtainable from (2.1) are satisfied.

(ii) On reducing the Fox H-function occurring in the left-hand side of (3.1) to the modified Bessel function of the second kind [6, p.18, Eq.(2.6.6)], we arrive at the following interesting inverse Laplace transform after a little simplification

$$L^{-1}\left\{s^{-(\rho+\frac{n_1}{2}+\frac{1}{2})}\prod_{i=1}^{\tau}(s^{l_i}+\alpha_i)^{-\sigma_i}H_{n_1}\left[\frac{\sqrt{s}}{2}\right]L_{(n_2)}^{(\alpha)}\left[\frac{1}{s}\right]K_v\left[\frac{z}{s}\right]\right\}$$

$$=2^{-3/2}z^{-1/2}\frac{t^{\rho+l_1\rho_1+\cdots+l_\tau\sigma_\tau-1}}{\Gamma(\sigma_1)\cdots\Gamma(\sigma_\tau)}\sum_{k_1=0}^{[n_1/2]}\sum_{k_2=0}^{n_2}\frac{(-n_1)_{2k_1}(-n_2)_{k_2}}{k_1!k_2!}(-1)^{k_1}\binom{n_2+\alpha}{n_2}$$

$$\frac{1}{(\alpha+1)_{k_2}} t^{k_1+k_2} H^{0,0:2,0;1,1;\cdots;1,1}_{0,1:0,2;1,1;\cdots;1,1}\left[\begin{matrix} zt\\ \alpha_1 t^{l_1}\\ \vdots \\ \alpha_\tau t^{l_\tau}\end{matrix}\right| \left.\begin{matrix} \text{——} : \text{——} ; (1-\sigma_1,1);\cdots;(1-\sigma_\tau,1) \\ (1-\rho-l_1\sigma_1-\cdots-l_\tau\sigma_\tau-k_1-k_2;1,l_1,\ldots,l_\tau) : (\frac{1}{4}\pm\frac{v}{2},\frac{1}{2});(0,1);\cdots;(0,1)\end{matrix}\right] \tag{3.2}$$

where $\mathrm{Re}(s)>0$, $z>0$, $\mathrm{Re}(\rho+l_1\sigma_1+\cdots+l_\tau\sigma_\tau\pm v+\frac{1}{2})>0$; $0<l_i<1$, $|\arg\alpha_i|<(1-l_i)\frac{\pi}{2}$, $i=1,\ldots,\tau$ or $l_1=l_2=\cdots=l_\tau=1$ and $\alpha_1>0,\ldots,\alpha_\tau>0$.

The inverse Laplace transform of the product of a wide variety of polynomials (which are special cases of $S_{n_1}^{m_1}$ and $S_{n_2}^{m_2}$) and numerous other simple special functions (which are particular cases of the Fox H-function) can also be obtained from our main result but we do not record them here for lack of space.

References

[1] C. Fox, *The G and H functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc. **98**(1961), 395-429.

[2] K. C. Gupta and R. C. Soni, *On the inverse Laplace transform*, Ganita Sandesh **4**(1990), 1-5.

[3] K. C. Gupta and R. C. Soni, *A unified inverse Laplace transform formula, functions of practical importance and H-functions*, J. Rajasthan Acad. Phy. Sci. **1**(2002), 7-16.

[4] A. M. Mathai and R. K. Saxena, *The H-function with applications in statistics and other disciplines*, Wiley Eastern Ltd., New Delhi, 1978.

[5] H. M. Srivastava, *A contour integral involving Fox's H-function*, Indian J. Math. **14**(1972), 1-6.

[6] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The H-functions of one and two variables with applications*, South Asian Publishers, New Delhi, 1982.

[7] H. M. Srivastava and R. Panda, *Some bilateral generating functions for a class of generalized hypergeometric polynomials*, J. Reine Angew. Math. **283/284**(1976), 265-274.

[8] H. M. Srivastava and N. P. Singh, *The integration of certain products of the multivariable H-function with a general class of polynomials*, Rend. Circ. Mat. Palermo **32**(1983), 157-187.

Department of Mathematics, M. N. Institute of Technology, Jaipur 302017, India.

E-mail: drrcsoni@yahoo.com