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INTEGRAL OPERATOR AND OSCILLATION OF SECOND ORDER ELLIPTIC EQUATIONS

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Abstract. By using integral operator, some oscillation criteria for second order elliptic differential equation

$$\sum_{i,j=1}^{d} D_i[A_{ij}(x)D_jy] + q(x)f(y) = 0, \quad x \in \Omega, \tag{E}$$

are established. The results obtained here can be regarded as the extension of the well-known Kamenev theorem to Eq.(E).

1. Introduction and Preliminaries

We are concerned with the oscillatory behavior of second order elliptic differential equation of the form

$$\sum_{i,j=1}^{d} D_i[A_{ij}(x)D_jy] + q(x)f(y) = 0, \tag{1}$$

where $x = (x_1, \cdots, x_d) \in \Omega(a) \subseteq \mathbb{R}^d$, $d \geq 2$, $|x| = [\sum_{i=1}^{d} x_i^2]^{1/2}$, $D_iy = \partial y/\partial x_i$ for all i, $\Omega(a) = \{x \in \mathbb{R}^d : |x| \geq a\}$ for some $a > 0$.

Throughout this paper, we assume that the following conditions holds:

(A_1) $A = (A_{ij})$ is a real symmetric positive definite matrix function with $A_{ij} \in C_{loc}^{1+\nu}(\Omega(a), \mathbb{R})$ for all i, j, and $\nu \in (0, 1)$.

Denote by $u_{\max}(x)$ the largest eigenvalue of the matrix A. We suppose that there exists a function $u \in C([a, \infty), \mathbb{R}^+)$ such that

$$u(r) \geq \max_{|x|=r} u_{\max}(x), \quad \text{for} \quad r \geq a;$$

(A_2) $q \in C_{loc}^{\nu}(\Omega(a), \mathbb{R})$, $\nu \in (0, 1)$, and $q(x)$ does not eventually vanish;

(A_2) $f \in C(\mathbb{R}, \mathbb{R}) \cup C^1(\mathbb{R} - \{0\}, \mathbb{R})$, $yf(y) > 0$ whenever $y \neq 0$, and $f'(y) \geq k > 0$ for all $y \neq 0$.

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As usual, a function $y \in C^{2+\nu}_{loc}(\Omega(a),\mathbb{R})$, $\nu \in (0,1)$, is called a solution of Eq.(1) if $y(x)$ satisfies Eq.(1) for all $x \in \Omega(a)$. We restrict our attention only the nontrivial solution of Eq.(1), i.e., to the solution $y(x)$ such that $\sup\{|y(x)| : x \in \Omega(b)\} > 0$ for every $b \geq a$. Regarding the question of existence of solution of Eq.(1) we refer the reader to the monograph [2]. A nontrivial solution $y(x)$ of Eq.(1) is said to be oscillatory in $\Omega(a)$ if the set $\{x \in \Omega(a) : y(x) = 0\}$ is unbounded, otherwise it is said to be nonoscillatory. Eq.(1) called oscillatory if all its nontrivial solutions are oscillatory.

Investigation of oscillation of Eq.(1) with variable coefficient $q(x)$ was initiated by Noussair and Swanson [5], who first extended the well-known Fite-Wintner Theorem (see Fite [1] and Wintner [10]) to Eq.(1). As an excellent survey paper, the reader is to recommended to Swanson [6]. Recently, Xu [7], [8] and Zhang et al. [9] have discussed the oscillation property of Eq.(1) under the assumption that the function $q(x)$ is an "integrally small" coefficient, and have shown that most of Kamenev's results in [3] do hold equally well in the case for Eq.(1). However, as far as the authors know there are few results by using integral averaging techniques [4], even through the function $q(x)$ is nonnegative. Motivated by this fact, we intend here to establish Kamenev's integral theorem for oscillation of Eq.(1) based on the integral operator [11]. We are especially interested in case where q may be take on negative values for arbitrarily large $|x|$. Our methodology is somewhat different from that of previous authors, we believe that our approach is simpler and also provides a more unified for the study of Kamenev-type oscillation theorems.

Now, we introduce the general mean and the integral operator.

Following Wong [11], let $D = \{(r,s) : r \geq s \geq a\} \in \mathbb{R}^2$, and $D_0 = \{(r,s) : r > s \geq a\} \in \mathbb{R}^2$. Consider the kernel function $H(r,s) \in C^1(D,\mathbb{R})$ such that the following conditions are satisfied:

(H_1) $H(r,r) = 0$ for $r \geq a$; $H(r,s) > 0$ on $(r,s) \in D_0$;

(H_2) For each $s \geq a$, $\lim_{r\to\infty} H(r,s) = \infty$, and there exists a positive constant k_0 such that

$$\lim_{r\to\infty} \frac{H(r,s)}{H(r,a)} = k_0, \quad \text{for all} \quad s \geq a;$$

(H_3) $\dfrac{\partial H}{\partial s}(r,s) \leq 0$, $-\dfrac{\partial H}{\partial s}(r,s) = h(r,s)H(r,s)$, and $\dfrac{\partial h}{\partial r}(r,s) \leq 0$, for $(r,s) \in D_0$.

Let $\rho \in C^1([a,\infty),\mathbb{R}^+)$, we define an integral operator Π^ρ_τ, which is defined in [11], in terms of $H(r,s)$ and $\rho(s)$ as

$$\Pi^\rho_\tau(\Theta; r) = \int_\tau^r H(r,s)\Theta(s)\rho(s)ds, \quad \text{for } r > \tau \geq a, \tag{2}$$

where $\Theta \in C([a,\infty),\mathbb{R})$.

For natational simplicity, for arbitrary given functions $\Phi \in C^1([a,\infty),\mathbb{R}^+)$ and $(u\phi) \in C^1([a,\infty),\mathbb{R})$, we define

$$Q(r) = \Phi(r)\left\{\int_{S_r} q(x)\,d\sigma + \frac{k}{\omega}u(r)\phi^2(r)r^{1-d} - [u(r)\phi(r)]'\right\},$$

$$p(r) = -\left[\frac{\Phi'(r)}{\Phi(r)} + \frac{2k}{\omega}\phi(r)r^{1-d}\right], \quad g(r) = \frac{kr^{1-d}}{\omega u(r)\Phi(r)}, \tag{3}$$

where $S_r = \{x \in \mathbb{R}^d : |x| = r\}$ for $r > a$, $d\sigma$ denotes the spherical integral element in $\mathbb{R}^d$, ω denotes the surface area of unit sphere in $\mathbb{R}^d$, i.e., $\omega = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$.

Lemma 1.[11] *Let $H(r,s) \in C^1(D,\mathbb{R})$ satisfying (H_1)–(H_3) and $\rho(s) \in C([a,\infty),\mathbb{R})$. Then*

(i) $\lim_{r\to\infty} \frac{1}{H(r,a)} \int_a^r H(r,s)\rho(s)\,ds = \infty$, *if* $\lim_{r\to\infty} \int_a^r \rho(s)\,ds = \infty$;

(ii) $\frac{1}{H(r,a)} \int_a^r H(r,s)\rho(s)\,ds$ *is nondecreasing in r, if $\rho(s) \geq 0$ on $[a,\infty)$.*

2. Main Results

In this section, we establish Kamenev's integral oscillation criteria for Eq.(1) based on the integral operator Π_τ^ρ. For the sake of simplicity, we always assume that the kernel function $H(r,s)$ satisfies condition $(H_1) - (H_3)$, the integral operator Π_τ^ρ and functions Q, p, g defined by (2) and (3), respectively.

Theorem 1. *Assume that there exist functions $\Phi \in C^1([a,\infty),\mathbb{R}^+)$ and $(u\phi) \in C^1([a,\infty),\ \mathbb{R})$ such that*

$$\limsup_{r\to\infty} \frac{1}{H(r,a)} \Pi_a^\tau \left(Q - \frac{1}{4g}\left[h + p - \frac{\rho'}{\rho}\right]^2 \right) = \infty. \tag{4}$$

Then Eq.(1) is oscillatory.

Proof. Let $y(x)$ be a nonoscillatory solution of Eq.(1). Without loss of generality, we may assume that $y(x) > 0$ for all $x \in \Omega(a)$. Define

$$W(x) = \frac{1}{f(y)}(A^\nabla y)(x), \tag{5}$$

where ${}^\nabla y = (D_1 y, D_2 y, \cdots, D_d y)^T$ denotes the gradient of $y(x)$. Differentiation (5) and using Eq.(1), we obtain that

$$\operatorname{div} W(x) = -q(x) - f'(y)(W^T A^{-1} W)(x).$$

Put

$$Z(r) = \Phi(r)\left[\int_{S_r} W(x)\cdot\mu(x)\,d\sigma + u(r)\phi(r)\right], \tag{6}$$

where $\mu(x) = x/|x|, (x \neq 0)$, denotes the outward unit normal. By Green's formula in (6), we have

$$Z'(r) = \frac{\Phi'(r)}{\Phi(r)} Z(r) + \Phi(r) \left\{ \int_{S_r} \operatorname{div} W(x)\, d\sigma + [u(r)\phi(r)]' \right\}$$
$$= \frac{\Phi'(r)}{\Phi(r)} Z(r) - \Phi(r) \left\{ \int_{S_r} q(x) d\sigma + \int_{S_r} f'(y)(W^T A^{-1} W)(x)\, d\sigma - [u(r)\phi(r)]' \right\}. \quad (7)$$

In view of (A_1),

$$(W^T A^{-1} W)(x) \geq u_{\max}^{-1}(x) |W(x)|^2.$$

The Schwartz inequality gives

$$\int_{S_r} |W(x)|^2\, d\sigma \geq \frac{r^{1-d}}{\omega} \left[\int_{S_r} W(x) \cdot \mu(x)\, d\sigma \right]^2.$$

Thus, by (6) and (7), we obtain

$$Z'(r) \leq \frac{\Phi'(r)}{\Phi(r)} Z(r) - \Phi(r) \left\{ \int_{S_r} q(x)\, d\sigma + \frac{k r^{1-d}}{\omega u(r)} \left[\int_{S_r} W(x) \cdot \mu(x)\, d\sigma \right]^2 - [u(r)\phi(r)]' \right\}$$
$$= \frac{\Phi'(r)}{\Phi(r)} Z(r) - \Phi(r) \left\{ \int_{S_r} q(x)\, d\sigma + \frac{k r^{1-d}}{\omega u(r)} \left[\frac{Z(r)}{\Phi(r)} - u(r)\phi(r) \right]^2 - [u(r)\phi(r)]' \right\}$$
$$= -\Phi(r) \left\{ \int_{S_r} q(x)\, d\sigma + \frac{k}{\omega} u(r) \phi^2(r) r^{1-d} - [u(r)\phi(r)]' \right\}$$
$$+ \left[\frac{\Phi'(r)}{\Phi(r)} + \frac{2k}{\omega} \phi(r) r^{1-d} \right] Z(r) - \frac{k r^{1-d}}{\omega u(r) \Phi(r)} Z^2(r). \quad (8)$$

Denote $V(r) = Z(r) + \frac{p(r)}{2g(r)}$, (8) can be rewritten as

$$Z'(r) + g(r) V^2(r) + Q(r) - \frac{1}{4} \frac{p^2(r)}{g(r)} \leq 0. \quad (9)$$

Applying the integral operator Π_b^ρ, $(b \geq a)$, to (9), we obtain

$$\Pi_b^\rho (g V^2) + \Pi_\tau^\rho \left(\left[h - \frac{\rho'}{\rho} \right] Z \right) + \Pi_\tau^\rho \left(Q - \frac{p^2}{4g} \right) \leq H(r,b) \rho(b) Z(b). \quad (10)$$

Completing squares of V in (10) yields

$$\Pi_b^\rho \left(g \left\{ V + \frac{1}{2g} \left[h - \frac{\rho'}{\rho} \right] \right\}^2 \right) + \Pi_b^\rho \left(Q - \frac{1}{4g} \left[h + p - \frac{\rho'}{\rho} \right]^2 \right) \leq H(r,b) \rho(b) Z(b). \quad (11)$$

Let $b = a$ and divide (11) through by $H(r,a)$. Note the first term is nonnegative, thus

$$\frac{1}{H(r,a)} \Pi_a^\tau \left(Q - \frac{1}{4g} \left[h + p - \frac{\rho'}{\rho} \right]^2 \right) \leq \rho(a) Z(a). \quad (12)$$

Take linsup in (12) as $r \to \infty$. Condition (4) gives a desired contradiction in (12). This proves Theorem 1.

Remark 1. If we choose the function $\phi(s) \equiv 0$, then Theorem 1 contains Theorem 4 in [5].

Theorem 2. *Let Φ and ϕ be as Theorem 1. Assume that there exist functions $\varphi_i \in C([a,\infty),\mathbb{R})$, $(i = 1,2)$, such that for $\tau \geq a$,*

$$\lim_{r\to\infty} \frac{1}{H(r,a)} \Pi_\tau^\rho \left(\frac{1}{g} \left[h + p - \frac{\rho'}{\rho} \right]^2 \right) \leq \varphi_1(\tau), \tag{13}$$

and

$$\limsup_{r\to\infty} \frac{1}{H(r,a)} \Pi_\tau^\rho (Q) \geq \varphi_2(\tau), \tag{14}$$

where φ_1 and φ_2 satisfy

$$\lim_{r\to\infty} \frac{1}{H(r,a)} \Pi_a^\rho \left(g\rho^{-2} \left[\varphi_2 - \frac{1}{4}\varphi_1 \right]_+^2 \right) = \infty, \tag{15}$$

and $[\varphi(r)]_+ = max\{\varphi(r), 0\}$. Then Eq.(1) is oscillatory.

Proof. Proceeding as in the proof Theorem 1, we see (11) holds for all $r \geq a$. Divide (11) by $H(r,a)$ and drop the nonnegative first term and obtain

$$\frac{1}{H(r,a)} \Pi_\tau^\rho(Q) - \frac{1}{4} \frac{1}{H(r,a)} \Pi_\tau^\rho \left(\frac{1}{g} \left[h + p - \frac{\rho'}{\rho} \right]^2 \right) \leq \frac{H(r,b)}{H(r,a)} \rho(\tau) Z(\tau).$$

Take limsup in above inequality as $r \to \infty$ and note from (13), (14) and (H_2) that

$$\varphi_2(\tau) - \frac{1}{4}\varphi_1(\tau) \leq k_0 \rho(\tau) Z(\tau).$$

From which it follows that

$$k_0^{-2} \rho^{-2}(\tau) \left[\varphi_2(\tau) - \frac{1}{4}\varphi_1(\tau) \right]_+^2 \leq Z^2(\tau). \tag{16}$$

Then, it follows from (15) and (16) that

$$\lim_{r\to\infty} \frac{1}{H(r,a)} \Pi_a^\rho (gZ^2) \geq \lim_{r\to\infty} \frac{k_0^{-2}}{H(r,a)} \Pi_a^\rho \left(g\rho^{-2} \left[\varphi_2 - \frac{1}{4}\varphi_1 \right]_+^2 \right) = \infty. \tag{17}$$

Next, we shall show that (17) is not possible.

For (11), set $b = a$. Dividing (11) through $H(r,a)$, and noting (13), (14), we have

$$\frac{1}{H(r,a)} \Pi_a^\rho \left(g \left\{ V + \frac{1}{2g} \left[h - \frac{\rho'}{\rho} \right] \right\}^2 \right) \leq \rho(a) Z(a) + \left[\varphi_2(b) - \frac{1}{4}\varphi_1(b) \right]. \tag{18}$$

We note that by Lemma 1 (ii), (18) and (13), the following limits exist and finite, that is

$$\lim_{r\to\infty}\frac{1}{H(r,a)}\Pi_a^\rho\left(g\left\{V+\frac{1}{2g}\left[h-\frac{\rho'}{\rho}\right]\right\}^2\right)<\infty,$$
$$\lim_{r\to\infty}\frac{1}{H(r,a)}\Pi_a^\rho\left(\frac{1}{g}\left[h+p-\frac{\rho'}{\rho}\right]^2\right)<\infty. \tag{19}$$

Noting that

$$Z(r)=V(r)-\frac{p(r)}{2g(r)}=\left(V+\frac{1}{2g}\left[h-\frac{\rho'}{\rho}\right]\right)-\frac{1}{2g}\left(h+p-\frac{\rho'}{\rho}\right).$$

Thus

$$g(s)Z^2(r)\le 2\left\{g(s)\left(V+\frac{1}{2g}\left[h-\frac{\rho'}{\rho}\right]\right)^2+\frac{1}{4g}\left(h+p-\frac{\rho'}{\rho}\right)^2\right\}.$$

Consequently, by (19), we obtain

$$\lim_{r\to\infty}\frac{1}{H(r,a)}\Pi_a^\rho(gZ^2)<\infty,$$

which contradicts (17). Therefore, the proof is complete.

Remark 2. Note that Theorem 1 and Theorem 2 do not require conditions $\int^\infty \frac{r^{1-d}}{u(r)}dr = \infty$, or $\int_{\Omega(b)} q(x)\,dx$, $(b\ge a)$, convergent in [5], [7-9].

In same way as it was done in [11], with an appropriate choice of the functions H and ρ, we can derive various oscillation criteria for Eq.(1) from Theorems 1 and 2. For example, define

$$H(r,s)=(r-s)^\alpha,\quad (r,s)\in D,\quad \alpha>1.$$

Corollary 1. *Let $\alpha>1$. Suppose that the function g is nondecreasing, and*

$$\limsup_{r\to\infty} r^{-\alpha}\int_a^r (r-s)^\alpha\left\{Q(s)-\frac{p^2(s)}{4g(s)}-\frac{1}{2}\left(\frac{p(s)}{g(s)}\right)'\right\}ds=\infty. \tag{20}$$

Then Eq.(1) is oscillatory.

Proof. Let $\rho(r)=1$ in Theorem 1, observe that

$$[p(s)+\alpha(r-s)^{-1}]^2=p^2(s)+2\alpha(r-s)^{-1}p(s)+\alpha^2(r-s)^{-2}.$$

By the Bonnet Theorem, for a fixed $r\ge a$ and some $\xi\in[a,r]$, we have

$$\int_a^r\frac{(r-s)^{\alpha-2}}{g(s)}\,ds=\frac{1}{g(a)}\int_a^\xi (r-s)^{\alpha-2}\,ds,$$

hence

$$\lim_{r\to\infty}\frac{1}{r^\alpha}\int_a^r\frac{(r-s)^{\alpha-2}}{g(s)}ds=0.$$

Using integrating by parts, we get

$$\alpha\int_a^r(r-s)^{\alpha-1}\frac{p(s)}{g(s)}\,ds=(r-a)^\alpha\frac{p(a)}{g(a)}+\int_a^r(r-s)^\alpha\left(\frac{p(s)}{g(s)}\right)'\,ds.$$

Thus, Corollary 1 from Theorem 1.

Now, let $\rho(r)=\exp(\int_a^r p(s)\,ds)$ in condition (4), we have

Corollary 2. *Let $\alpha>1$. If*

$$\limsup_{r\to\infty} r^{-\alpha}\int_a^r(r-s)^\alpha Q(s)\exp\left(\int_a^s p(s)\,d\tau\right)ds=\infty, \tag{21}$$

and

$$\lim_{r\to\infty} r^{-\alpha}\int_a^r(r-s)^{\alpha-2}[g(s)]^{-1}\exp\left[\int_a^s p(\tau)\,d\tau\right]ds<\infty. \tag{22}$$

Then Eq.(1) *is oscillatory.*

Remark 3. With the same choice of the functions $H(r,s)$ and $\rho(s)$ as corollary 1 and Corollary 2, more general oscillation criteria for Eq.(1) can be obtained by Theorem 2 similarly. Here we omit the details.

Example 1. Consider the linear elliptic equation of second order

$$\Delta y+(|x|^{\sigma-1}\sin|x|)y=0, \tag{23}$$

where $x\in\Omega(1)$, $d=2$, $\sigma>0$, $u(r)=1$, $q(x)=|x|^{\sigma-1}\sin|x|$, $f'(y)=1$ for all $|x|>1$. Choose $\Phi(r)=1$, $\varphi(r)=0$, then

$$Q(r)=2\pi r^\sigma\sin r,\quad g(r)=(2\pi r)^{-1},\quad p(s)=0.$$

For $\sigma>0$, it is easy to verify that

$$\limsup_{r\to\infty}\frac{1}{r^2}\int_1^r(r-s)^2Q(s)ds=\limsup_{r\to\infty}\frac{1}{r^2}\int_1^r(r-s)^2 s^\sigma\sin s\,ds=\infty,$$

and

$$\lim_{r\to\infty}\frac{1}{r^2}\int_1^r\frac{1}{g(s)}ds=\pi.$$

Hence, Eq.(23) is oscillatory by Corollary 2.

Example 2. Consider the nonlinear elliptic equation of second order

$$\Delta y+(|x|^\upsilon\cos|x|+\sin|x|)(y+y^3)=0, \tag{24}$$

where $x \in \Omega(1)$, $d = 2$, υ is a constant with $2\upsilon > -1$, $u(r) = 1$, $q(x) = |x|^\upsilon \cos|x| + \sin|x|$, and $f(y) = y + y^3$. Let $\Phi(r) = r^{-1}$, $\varphi(r) = 0$, then

$$g(r) = (2\pi)^{-1}, \quad p(r) = r^{-1}, \quad Q(r) = 2\pi(r^\upsilon \cos r + \sin r), \quad f'(y) = 1 + y^2.$$

Taking $H(r, s) = (r - s)^2$ for $(r, s) \in D$ and $\rho(r) = 1$, we have, for all $\tau \geq 1$,

$$\limsup_{r\to\infty} \frac{1}{r^2} \int_\tau^r (r - s)^2 Q(s)\, ds \geq -2\pi\tau^\upsilon \cos\tau - \varepsilon,$$

and

$$\lim_{r\to\infty} \frac{1}{r^2} \int_\tau^r \frac{1}{g(s)} (r - s)^2 [2(r - s)^{-1} + s^{-1}]^2\, ds = \frac{2\pi}{\tau},$$

where ε is a positive constant.

Now, set $\varphi_1(\tau) = (2\pi\tau)^{-1}$ and $\varphi_2(\tau) = -2\pi\tau^\upsilon \cos\tau - \varepsilon$. Then, there exists an integer N such that $(2N + 1)\pi - \frac{\pi}{4} > 1$ and if $n \geq N$,

$$(2n + 1)\pi - \frac{\pi}{4} \leq \tau \leq (2n + 1)\pi + \frac{\pi}{4}, \quad \varphi_2(\tau) - \frac{1}{4}\varphi_1(\tau) \geq \delta\tau^\upsilon,$$

where δ is a small constant. Taking into account that $2\upsilon > -1$, we obtain

$$\begin{aligned}\lim_{r\to\infty} \int_a^r \left[\varphi_2(\tau) - \frac{1}{4}\varphi_1(\tau)\right]_+^2 ds &\geq \sum_{n=N}^{\infty} \delta^2 \int_{(2n+1)\pi-\frac{\pi}{4}}^{(2n+1)\pi+\frac{\pi}{4}} s^{2\upsilon}\, ds \\ &\geq \sum_{n+N}^{\infty} \delta^2 \int_{(2n+1)\pi-\frac{\pi}{4}}^{(2n+1)\pi+\frac{\pi}{4}} s^{-1}\, ds = \infty.\end{aligned}$$

This implies that, by Lemma 1 (i),

$$\lim_{r\to\infty} \frac{1}{r^2} \int_a^r \frac{1}{2\pi} (r - s)^2 [\varphi_2(\tau) - \frac{1}{4}\varphi_1(\tau)]_+^2\, ds = \infty.$$

Accordingly, all conditions of Theorem 2 are satisfied, and hence, Eq.(24) is oscillatory.

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