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# SOME HERMITE-HADAMARD TYPE INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL CALCULUS

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**Abstract**. We provide some new Hermite-Hadamard type inequalities for functions whose derivatives in absolute value are convex, via Riemann-Liouville fractional integration.

## 1. Introduction

The Hermite-Hadamard inequality asserts that for every convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ one has

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. Both inequalities hold in reversed direction if $f$ is concave.

A remarkable variety of refinements and generalizations of Hermite-Hadamard inequality involving convex functions in the second sense have been found; see, for example, [1], [3], [5] and the references therein.

Our aim is to establish, using the Riemann-Liouville fractional calculus, some new Hermite-Hadamard type inequalities. We shall deal with functions whose derivatives in absolute value are convex.

Let $f \in L^1[a,b]$, where $a \ge 0$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$, of order $\alpha > 0$, are defined by

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \text{ for } x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \text{ for } x < b,$$

respectively. Here, $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\mathrm{d}t$ is the Gamma function. We also make the convention

$$J^0_{a+}f(x) = J^0_{b-}f(x) = f(x).$$

More details about the Riemann-Liouville fractional integrals may be found in [2].

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## 2. A lemma

We assume throughout the present paper that $[a,b]$ is a subinterval of $[0,\infty)$ and $f:[a,b]\to\mathbb{R}$ is a function differentiable on $(a,b)$ such that $f'\in L^1[a,b]$. Before stating the results we establish the notation. Throughout this paper we denote the *cumulative to the left* $\alpha-$*gap* by

$$\mathscr{L}_\alpha(a,b)=\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\frac{3a+b}{4}-}f(a)+J^\alpha_{\frac{a+b}{2}-}f\left(\frac{3a+b}{4}\right)\right.$$
$$\left.+J^\alpha_{\frac{a+3b}{4}-}f\left(\frac{a+b}{2}\right)+J^\alpha_{b-}f\left(\frac{a+3b}{4}\right)\right].$$

Thus, the particular case $\alpha=1$ gives

$$\mathscr{L}(a,b)=\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

In order to prove our main results we need the following lemma.

**Lemma 1.** *We have*

$$\mathscr{L}_\alpha(a,b)=\frac{b-a}{16}\left[\int_0^1 t^\alpha f'\left(t\frac{3a+b}{4}+(1-t)a\right)\mathrm{d}t+\int_0^1\left(t^\alpha-1\right)f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t\right.$$
$$\left.+\int_0^1 t^\alpha f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\mathrm{d}t+\int_0^1\left(t^\alpha-1\right)f'\left(tb+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t\right].$$

**Proof.** We use the integration by parts and appropriate substitutions (such as $u=\frac{3a+b}{4}t+a(1-t)$, $v=\frac{a+b}{2}t+\frac{3a+b}{4}(1-t)$, ...) to show that

$$\frac{b-a}{16}\int_0^1 t^\alpha f'\left(t\frac{3a+b}{4}+(1-t)a\right)\mathrm{d}t=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\frac{3a+b}{4}-}f(a),$$

$$\frac{b-a}{16}\int_0^1\left(t^\alpha-1\right)f'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\mathrm{d}t=\frac{1}{4}f\left(\frac{3a+b}{4}\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\frac{a+b}{2}-}f\left(\frac{3a+b}{4}\right),$$

$$\frac{b-a}{16}\int_0^1 t^\alpha f'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\mathrm{d}t=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{\frac{a+3b}{4}-}f\left(\frac{a+b}{2}\right),$$

$$\frac{b-a}{16}\int_0^1\left(t^\alpha-1\right)f'\left(tb+(1-t)\frac{a+3b}{4}\right)\mathrm{d}t=\frac{1}{4}f\left(\frac{a+3b}{4}\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}J^\alpha_{b-}f\left(\frac{a+3b}{4}\right).$$

The proof is completed. □

## 3. Inequalities of Hermite-Hadamard type

We are now in a position to state and prove the following:

**Theorem 1.** *Assume $|f'|$ is convex on $[a,b]$. Then*

$$\begin{aligned}|\mathscr{L}_\alpha(a,b)| \leq \frac{b-a}{32(\alpha+1)(\alpha+2)} &\left[2\left|f'(a)\right| + \left(\alpha^2+5\alpha+2\right)\left|f'\left(\frac{3a+b}{4}\right)\right|\right.\\ &\left.+\left(\alpha^2+\alpha+2\right)\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\alpha^2+5\alpha+2\right)\left|f'\left(\frac{a+3b}{4}\right)\right| + \alpha(\alpha+1)\left|f'(b)\right|\right].\end{aligned}$$

**Proof.** Using Lemma 1 and taking modulus, we infer from the convexity of $\left|f'\right|$ that

$$\begin{aligned}|\mathscr{L}_\alpha(a,b)| \leq \frac{b-a}{16}&\left\{\int_0^1 t^\alpha\left[t\left|f'\left(\frac{3a+b}{4}\right)\right| + (1-t)\left|f'(a)\right|\right]\mathrm{d}t\right.\\ &+\int_0^1\left(1-t^\alpha\right)\left[t\left|f'\left(\frac{a+b}{2}\right)\right| + (1-t)\left|f'\left(\frac{3a+b}{4}\right)\right|\right]\mathrm{d}t\\ &+\int_0^1 t^\alpha\left[t\left|f'\left(\frac{a+3b}{4}\right)\right| + (1-t)\left|f'\left(\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\\ &\left.+\int_0^1\left(1-t^\alpha\right)\left[t\left|f'(b)\right| + (1-t)\left|f'\left(\frac{a+3b}{4}\right)\right|\right]\mathrm{d}t\right\}.\end{aligned}$$

The result follows after a straightforward computation in the right hand side term. This ends the proof. □

We recall that the Beta function (the Euler integral of the first kind), is defined by

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t$$

for $x,y>0$.
Our next results reads as:

**Theorem 2.** *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$. Then*

$$\begin{aligned}|\mathscr{L}_\alpha(a,b)| \leq \frac{b-a}{16}\left(\frac{1}{2}\right)^{1/q}\left(\frac{1}{\alpha}\right)^{1/p}&\left\{\left(\frac{\alpha}{\alpha p+1}\right)^{1/p}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q + \left|f'(a)\right|^q\right]^{1/q}\right.\\ &+\left[\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right]^{1/p}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q + \left|f'\left(\frac{3a+b}{4}\right)\right|^q\right]^{1/q}\\ &+\left(\frac{\alpha}{\alpha p+1}\right)^{1/p}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\\ &\left.+\left[\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right]^{1/p}\left[\left|f'\left(\frac{a+3b}{4}\right)\right|^q + \left|f'(b)\right|^q\right]^{1/q}\right\},\end{aligned}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

We denote

$$I_1 = \int_0^1\left|f'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q\mathrm{d}t,$$

$$I_2 = \int_0^1 \left| f'\left( t\frac{a+b}{2} + (1-t)\frac{3a+b}{4} \right) \right|^q \mathrm{d}t,$$

$$I_3 = \int_0^1 \left| f'\left( t\frac{a+3b}{4} + (1-t)\frac{a+b}{2} \right) \right|^q \mathrm{d}t$$

and

$$I_4 = \int_0^1 \left| f'\left( tb + (1-t)\frac{a+3b}{4} \right) \right|^q \mathrm{d}t.$$

**Proof.** According to Lemma 1 and Hölder's inequality, we have

$$\begin{aligned} |\mathscr{L}_\alpha(a,b)| \le \frac{b-a}{16} \Bigg[ & \left( \int_0^1 (t^\alpha)^p \mathrm{d}t \right)^{1/p} \left( (I_1)^{1/q} + (I_3)^{1/q} \right) \\ & + \left( \int_0^1 (1-t^\alpha)^p \mathrm{d}t \right)^{1/p} \left( (I_2)^{1/q} + (I_4)^{1/q} \right) \Bigg]. \end{aligned}$$

Here

$$\begin{aligned} I_1 &\le \left| f'\left( \frac{3a+b}{4} \right) \right|^q \int_0^1 t\mathrm{d}t + \left| f'(a) \right|^q \int_0^1 (1-t)\mathrm{d}t \\ &= \frac{1}{2} \left| f'\left( \frac{3a+b}{4} \right) \right|^q + \frac{1}{2} \left| f'(a) \right|^q, \\ I_2 &\le \frac{1}{2} \left| f'\left( \frac{a+b}{2} \right) \right|^q + \frac{1}{2} \left| f'\left( \frac{3a+b}{4} \right) \right|^q, \\ I_3 &\le \frac{1}{2} \left| f'\left( \frac{a+3b}{4} \right) \right|^q + \frac{1}{2} \left| f'\left( \frac{a+b}{2} \right) \right|^q \end{aligned}$$

and

$$I_4 \le \frac{1}{2} \left| f'(b) \right|^q + \frac{1}{2} \left| f'\left( \frac{a+3b}{4} \right) \right|^q.$$

These last inequalities hold due to the convexity of $\left| f' \right|^q$ on $[a,b]$. The proof is complete. □

**Theorem 3.** *Assume $|f'|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$. Then the following inequality holds:*

$$\begin{aligned} |\mathscr{L}_\alpha(a,b)| \le \frac{b-a}{16} & \left( \frac{1}{\alpha+1} \right)^{1/p} \left( \frac{1}{2(\alpha+1)(\alpha+2)} \right)^{1/q} \\ & \times \Bigg\{ \left[ 2(\alpha+1) \left| f'\left( \frac{3a+b}{4} \right) \right|^q + 2 \left| f'(a) \right|^q \right]^{1/q} \\ & + \alpha^{1/p} \left[ \alpha(\alpha+1) \left| f'\left( \frac{a+b}{2} \right) \right|^q + (\alpha^2+3\alpha) \left| f'\left( \frac{3a+b}{4} \right) \right|^q \right]^{1/q} \\ & + \left[ 2(\alpha+1) \left| f'\left( \frac{a+3b}{4} \right) \right|^q + 2 \left| f'\left( \frac{a+b}{2} \right) \right|^q \right]^{1/q} \\ & + \alpha^{1/p} \left[ \alpha(\alpha+1) \left| f'(b) \right|^q + (\alpha^2+3\alpha) \left| f'\left( \frac{a+3b}{4} \right) \right|^q \right]^{1/q} \Bigg\}. \end{aligned}$$

**Proof.** Using Lemma 1, the convexity of $|f'|^q$ on $[a,b]$ and the power-mean inequality, we have

$$|\mathcal{L}_\alpha(a,b)| \le \frac{b-a}{16}\left[\left(\int_0^1 t^\alpha \mathrm{d}t\right)^{1/p}\left((J_1)^{1/q}+(J_3)^{1/q}\right)\right.$$
$$\left.+\left(\int_0^1 \left(1-t^\alpha\right)\mathrm{d}t\right)^{1/p}\left((J_2)^{1/q}+(J_4)^{1/q}\right)\right],$$

where

$$\begin{aligned}
J_1 &= \int_0^1 t^\alpha \left|f'\left(t\frac{3a+b}{4}+(1-t)\,a\right)\right|^q \mathrm{d}t \\
&\le \frac{1}{\alpha+2}\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'(a)\right|^q, \\
J_2 &= \int_0^1 \left(1-t^\alpha\right)\left|f'\left(t\frac{a+b}{2}+(1-t)\,\frac{3a+b}{4}\right)\right|^q \mathrm{d}t \\
&\le \frac{\alpha}{2(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{3a+b}{4}\right)\right|^q, \\
J_3 &= \int_0^1 t^\alpha \left|f'\left(t\frac{a+3b}{4}+(1-t)\,\frac{a+b}{2}\right)\right|^q \mathrm{d}t \\
&\le \frac{1}{\alpha+2}\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\frac{1}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q
\end{aligned}$$

and

$$\begin{aligned}
J_4 &= \int_0^1 \left(1-t^\alpha\right)\left|f'\left(tb+(1-t)\,\frac{a+3b}{4}\right)\right|^q \mathrm{d}t \\
&\le \frac{\alpha}{2(\alpha+2)}\left|f'(b)\right|^q+\frac{\alpha^2+3\alpha}{2(\alpha+1)(\alpha+2)}\left|f'\left(\frac{a+3b}{4}\right)\right|^q.
\end{aligned}$$

Hence the proof of the theorem is complete. □

**Theorem 4.** *Assume $|f'|^q$ is concave on $[a,b]$ for some fixed $q>1$. Then*

$$|\mathcal{L}_\alpha(a,b)| \le \frac{b-a}{16}\left[\left(\frac{1}{\alpha p+1}\right)^{1/p}\left|f'\left(\frac{7a+b}{8}\right)\right|+\left(\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right)^{1/p}\left|f'\left(\frac{5a+3b}{8}\right)\right|\right.$$
$$\left.+\left(\frac{1}{\alpha p+1}\right)^{1/p}\left|f'\left(\frac{3a+5b}{8}\right)\right|+\left(\frac{1}{\alpha}\mathrm{B}\left(p+1,\frac{1}{\alpha}\right)\right)^{1/p}\left|f'\left(\frac{a+7b}{8}\right)\right|\right\},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**Proof.** From Lemma 1 and Hölder's integral inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$|\mathcal{L}_\alpha(a,b)| \le \frac{b-a}{16}\left[\left(\int_0^1 \left(t^\alpha\right)^p \mathrm{d}t\right)^{1/p}\left((I_1)^{1/q}+(I_3)^{1/q}\right)\right.$$
$$\left.+\left(\int_0^1 \left(1-t^\alpha\right)^p \mathrm{d}t\right)^{1/p}\left((I_2)^{1/q}+(I_4)^{1/q}\right)\right]$$

for all $x \in [a,b]$, where

$$I_1 \leq \left| f'\left( \frac{\frac{3a+b}{4}+a}{2} \right) \right|^q = \left| f'\left( \frac{7a+b}{8} \right) \right|^q,$$

$$I_2 \leq \left| f'\left( \frac{5a+3b}{8} \right) \right|^q,$$

$$I_3 \leq \left| f'\left( \frac{3a+5b}{8} \right) \right|^q$$

and

$$I_4 \leq \left| f'\left( \frac{a+7b}{8} \right) \right|^q.$$

We used the concavity of $|f'|^q$ on $[a,b]$ and the inequality (1.1) in order to obtain the last four inequalities. This completes the proof of the theorem. □

## 4. Remarks

**Remark 1.** For $\alpha = 1$ in the Theorems 1, 2, 3, respectively 4, we recover the results stated in ([4, Theorems 1-4]). Also for $\alpha = 1$ in Lemma 1, we get ([4, Lemma 1]).

**Remark 2.** By considering the *cumulative to the right $\alpha-$gap* defined as

$$\mathscr{R}_\alpha(a,b) = -\frac{1}{2}\left[ \frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] + \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[ J_{a+}^{\alpha} f\left(\frac{3a+b}{4}\right) + J_{\frac{3a+b}{4}+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{\frac{a+b}{2}+}^{\alpha} f\left(\frac{a+3b}{4}\right) + J_{\frac{a+3b}{4}+}^{\alpha} f(b)\right],$$

one can obtain similar results. However, this is not the purpose of the present paper.

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