



AN EXTENSION OF A BASIC UNIVALENCE CRITERION

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Abstract. Some sufficient conditions for univalence and quasiconformal extension of a class of functions defined by an integral operator are discussed with some examples. This condition involves two arbitrary functions g and h analytic in the unit disk. A number of well-known univalent conditions would follow upon specializing the functions and the parameters involved in our main result.

1. Introduction

Let $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r, 0 < r \leq 1\}$ be the disk of radius r centered at the origin and let $\mathcal{U} = \mathcal{U}_1$ be the open unit disk.

Denote by \mathcal{A} the class of analytic functions in \mathcal{U} which satisfy the usual normalization $f(0) = f'(0) - 1 = 0$.

The first results concerning univalence criteria are related to the univalence of an analytic function in the open unit disk \mathcal{U} . Among the most important sufficient conditions for univalence, we mention those obtained by Nehari [13], Goluzin [8], Ozaki and Nunokawa [17], Becker [3], and by Lewandowski [12].

For examples, Becker [3] has given the following condition

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathcal{U}) \quad (1.1)$$

for $f \in \mathcal{A}$ to be univalent in \mathcal{U} .

Furthermore, Ozaki and Nunokawa [17] have showed the following condition

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 1 \quad (z \in \mathcal{U}) \quad (1.2)$$

for $f \in \mathcal{A}$ to be univalent in \mathcal{U} .

Some extensions of these univalence criteria for an integral operator were obtained in the

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papers [5], [7], [14], [15], [16], [18], and [24]. Tudor [24] has discussed the condition for the integral operator $F_\alpha(z)$ defined by

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u) du \right)^{\frac{1}{\alpha}} \tag{1.3}$$

to be univalent in \mathcal{U} .

From the main result of this paper, would follow all the univalence criteria mentioned above and in the same time other new ones. Recently, Deniz and Orhan [6] gave some results for univalence of functions $f \in \mathcal{A}$, but not for our integral operator of $f \in \mathcal{A}$.

2. Loewner chains and quasiconformal extensions

Before proving our main result, we need a brief summary of theory of Loewner chains.

A function $L(z, t) : \mathcal{U} \times [0, \infty) \rightarrow \mathbb{C}$ is said to be a *Loewner chain* or a *subordination chain* if:

- (i) $L(z, t)$ is analytic and univalent in \mathcal{U} for all $t \geq 0$.
- (ii) $L(z, t) \prec L(z, s)$ for all $0 \leq t \leq s < \infty$, where the symbol " \prec " stands for subordination.

The following result due to Pommerenke is often used to obtain univalence criteria.

Theorem 2.1 ([20, 21]). *Let $L(z, t) = a_1(t)z + \dots$ be an analytic function in \mathcal{U}_r ($0 < r \leq 1$) for all $t \geq 0$. Suppose that:*

- (i) $L(z, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$, locally uniform with respect to $z \in \mathcal{U}_r$.
- (ii) $a_1(t)$ is a complex valued continuous function on $[0, \infty)$ such that $a_1(t) \neq 0$, $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ and

$$\left\{ \frac{L(z, t)}{a_1(t)} \right\}_{t \geq 0}$$

is a normal family of functions in \mathcal{U}_r .

- (iii) There exists an analytic function $p : \mathcal{U}_r \times [0, \infty) \rightarrow \mathbb{C}$ satisfying $\Re p(z, t) > 0$ for all $(z, t) \in \mathcal{U} \times [0, \infty)$ and

$$z \frac{\partial L(z, t)}{\partial z} = p(z, t) \frac{\partial L(z, t)}{\partial t}, z \in \mathcal{U}_r, a.e t \geq 0. \tag{2.1}$$

Then, for each $t \geq 0$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk \mathcal{U} , i.e $L(z, t)$ is a Loewner chain.

Let k be a constant in $[0, 1)$. Recall that a homeomorphism f of $G \subset \mathbb{C}$ is said to be *k-quasiconformal* if $\partial_z f$ and $\partial_{\bar{z}} f$ are locally integrable on G and satisfy $|\partial_{\bar{z}} f| \leq k |\partial_z f|$ almost everywhere in G .

The method of constructing quasiconformal extension criteria is based on the following result due to Becker (see [3], [4]).

Theorem 2.2. *Suppose that $L(z, t)$ is a subordination chain. Consider*

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}, \quad z \in \mathcal{U}, \quad t \geq 0$$

where $p(z, t)$ is defined by (2.1). If

$$|w(z, t)| \leq k, \quad 0 \leq k < 1$$

for all $z \in \mathcal{U}$ and $t \geq 0$, then $L(z, t)$ admits a continuous extension to $\bar{\mathcal{U}}$ for each $t \geq 0$ and the function $F(z, \bar{z})$ defined by

$$F(z, \bar{z}) = \begin{cases} L(z, 0) & , \text{if } |z| < 1 \\ L\left(\frac{z}{|z|}, \log|z|\right) & , \text{if } |z| \geq 1 \end{cases}$$

is a k -quasiconformal extension of $L(z, 0)$ to \mathbb{C} .

Examples of quasiconformal extension criteria can be found in [1], [2], [19] and more recently in [9], [10], [11].

3. Univalence criteria

Making use of Theorem 2.1 we can prove our main result.

Theorem 3.1. *Let α be a complex number, $\Re\alpha > 0$, $m \in \mathbb{R}$, $m > 0$ and $f \in \mathcal{A}$. If there exist two analytic functions in \mathcal{U} , $g(z) = 1 + b_1z + \dots$ and $h(z) = c_0 + c_1z + \dots$ such that the inequalities*

$$\left| \left(\frac{f'(z)}{g(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{3.1}$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{g(z)} - 1 \right) |z|^{\alpha(m+1)} + \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \left(2\alpha \frac{zf'(z)h(z)}{g(z)} + \frac{zg'(z)}{g(z)} \right) \right. \\ & \left. + \frac{(1 - |z|^{\alpha(m+1)})^2}{\alpha|z|^{\alpha(m+1)}} z^2 \left[\alpha \frac{f'(z)h^2(z)}{g(z)} + \frac{g'(z)h(z)}{g(z)} + (\alpha - 1) \frac{h(z)}{z} - h'(z) \right] - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \end{aligned} \tag{3.2}$$

hold true for all $z \in \mathcal{U} \setminus \{0\}$, then the function

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u) du \right)^{1/\alpha} \tag{3.3}$$

is analytic and univalent in \mathcal{U} , where the principal branch is intended.

Proof. Let a be a positive real number and let the function $h_1(z, t)$ be defined by

$$h_1(z, t) = 1 + (e^{(m+1)axt} - 1)e^{-at}zh(e^{-at}z).$$

For all $t \geq 0$ and $z \in \mathcal{U}$, we have $e^{-at}z \in \mathcal{U}$ and, from the analyticity of h in \mathcal{U} , it follows that $h_1(z, t)$ is also analytic in \mathcal{U} . Since $h_1(0, t) = 1$, there exists a disk \mathcal{U}_{r_1} , $0 < r_1 < 1$ in which $h_1(z, t) \neq 0$ for all $t \geq 0$. Let us define the function $h_2(z, t)$ by

$$h_2(z, t) = \alpha \int_0^{e^{-at}z} u^{\alpha-1} f'(u) du.$$

Letting

$$h_3(z, t) = e^{-aat} \left(1 + \frac{\alpha}{\alpha+1} e^{-at} z + \dots \right),$$

we have that $h_2(z, t) = z^\alpha h_3(z, t)$. It can be shown easily that $h_3(z, t)$ is analytic in \mathcal{U}_{r_1} and $h_3(0, t) = e^{-aat}$. It follows that the function

$$h_4(z, t) = h_3(z, t) + (e^{maat} - e^{-aat}) \frac{g(e^{-at}z)}{h_1(z, t)}$$

is also analytic in a disk \mathcal{U}_{r_2} , $0 < r_2 \leq r_1$ and $h_4(0, t) = e^{maat}$. Therefore, there is a disk \mathcal{U}_{r_3} , $0 < r_3 \leq r_2$ in which $h_4(z, t) \neq 0$, for all $t \geq 0$ and thus, we can choose an analytic branch of $[h_4(z, t)]^{1/\alpha}$, denoted by $h_5(z, t)$. We choose the branch which is equal to e^{mat} at the origin. From these reasons, it follows that the function

$$L(z, t) = zh_5(z, t) = e^{mat} z + a_2(t)z^2 + \dots$$

is analytic in \mathcal{U}_{r_3} for all $t \geq 0$. It is easy to see that the function $L(z, t)$ can be also written in the form

$$L(z, t) = \left[\alpha \int_0^{e^{-at}z} u^{\alpha-1} f'(u) du + \frac{(e^{maat} - e^{-aat})z^\alpha g(e^{-at}z)}{1 + (e^{(m+1)aat} - 1)e^{-at}zh(e^{-at}z)} \right]^{\frac{1}{\alpha}}. \quad (3.4)$$

If $L(z, t) = a_1(t)z + a_2(t)z^2 + \dots$ is the Taylor expansion of $L(z, t)$ in \mathcal{U}_{r_3} , then we have $a_1(t) = e^{mat}$ and therefore $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$.

From the analyticity of $L(z, t)$ in \mathcal{U}_{r_3} , it follows that there exists a number r_4 , $0 < r_4 \leq r_3$, and a constant $K = K(r_4)$ such that

$$|L(z, t)/a_1(t)| < K, \quad \forall z \in \mathcal{U}_{r_4}, \quad t \geq 0,$$

and thus $\{L(z, t)/a_1(t)\}$ is a normal family in \mathcal{U}_{r_4} . From the analyticity of $\partial L(z, t)/\partial t$, for all fixed numbers $T > 0$ and r_5 , $0 < r_5 \leq r_4$, there exists a constant $K_1 > 0$ (that depends on T and r_5) such that

$$\left| \frac{\partial L(z, t)}{\partial t} \right| < K_1, \quad \forall z \in \mathcal{U}_{r_5}, \quad t \in [0, T].$$

It follows that the function $L(z, t)$ is locally absolutely continuous in $[0, \infty)$, locally uniform with respect to $z \in \mathcal{U}_{r_5}$. The function $p(z, t)$ defined by

$$p(z, t) = z \frac{\partial L(z, t)}{\partial z} \Big/ \frac{\partial L(z, t)}{\partial t}$$

is analytic in a disk $\mathcal{U}_r, 0 < r \leq r_5$, for all $t \geq 0$.

In order to prove that the function $p(z, t)$ is analytic and has positive real part in \mathcal{U} , we will show that the function

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}, z \in \mathcal{U}_r, t \geq 0$$

has an analytic extension in \mathcal{U} and

$$|w(z, t)| < 1 \text{ for all } z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.5}$$

An elementary calculation gives that

$$w(z, t) = \frac{(1 + a)\mathcal{G}(z, t) + 1 - ma}{(1 - a)\mathcal{G}(z, t) + 1 + ma}, \tag{3.6}$$

where $\mathcal{G}(z, t)$ is given by

$$\begin{aligned} \mathcal{G}(z, t) = & \left(\frac{f'(e^{-at}z)}{g(e^{-at}z)} - 1 \right) e^{-(1+m)aat} \\ & + \frac{1 - e^{-(1+m)aat}}{\alpha} \left[2\alpha \frac{e^{-at}z f'(e^{-at}z) h(e^{-at}z)}{g(e^{-at}z)} + \frac{e^{-at}z g'(e^{-at}z)}{g(e^{-at}z)} \right] \\ & + \frac{(1 - e^{-(1+m)aat})^2}{\alpha e^{-(1+m)aat}} \left[\alpha \frac{e^{-2at}z^2 f'(e^{-at}z) h^2(e^{-at}z)}{g(e^{-at}z)} + \frac{e^{-2at}z^2 g'(e^{-at}z) h(e^{-at}z)}{g(e^{-at}z)} \right. \\ & \left. + (\alpha - 1)e^{-at}zh(e^{-at}z) - e^{-2at}z^2 h'(e^{-at}z) \right]. \end{aligned} \tag{3.7}$$

It is easy to prove that the condition (3.5) is equivalent to

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| < \frac{m+1}{2} \text{ for all } z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.8}$$

For $t = 0$ and $z \neq 0$, in view of (3.1), we get

$$\left| \mathcal{G}(z, 0) - \frac{m-1}{2} \right| = \left| \left(\frac{f'(z)}{g(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2}. \tag{3.9}$$

Also, since m is a positive number, the following inequality

$$\left| \mathcal{G}(0, t) - \frac{m-1}{2} \right| = \left| \frac{m-1}{2} \right| < \frac{m+1}{2}, \quad t > 0 \tag{3.10}$$

is true. Let t be a fixed number, $t > 0$ and let $z \in \mathcal{U}, z \neq 0$. Since $|e^{-at}z| \leq e^{-at} < 1$ for all $z \in \overline{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$, from (3.7), we conclude that the function $\mathcal{G}(z, t)$ is analytic in $\overline{\mathcal{U}}$. Using the maximum modulus principle, it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|\mathcal{G}(z, t)| < \max_{|\xi|=1} |\mathcal{G}(\xi, t)| = |\mathcal{G}(e^{i\theta}, t)|. \tag{3.11}$$

Denote $u = e^{-at}e^{i\theta}$. Then $|u| = e^{-at} < 1$, $e^{(m+1)at} = 1/|u|^{m+1}$ and therefore

$$\begin{aligned} \mathcal{G}(e^{i\theta}, t) &= \left(\frac{f'(u)}{g(u)} - 1\right) |u|^{\alpha(m+1)} + \frac{1 - |u|^{\alpha(m+1)}}{\alpha} \left(2\alpha \frac{uf'(u)h(u)}{g(u)} + \frac{ug'(u)}{g(u)}\right) \\ &\quad + \frac{(1 - |u|^{\alpha(m+1)})^2}{\alpha|u|^{\alpha(m+1)}} u^2 \left[\alpha \frac{f'(u)h^2(u)}{g(u)} + \frac{g'(u)h(u)}{g(u)} + (\alpha - 1) \frac{h(u)}{u} - h'(u)\right]. \end{aligned}$$

Since $u \in \mathcal{U}$, the inequality (3.2) implies

$$\left| \mathcal{G}(e^{i\theta}, t) - \frac{m-1}{2} \right| \leq \frac{m+1}{2}. \tag{3.12}$$

From (3.9), (3.10), (3.11) and (3.12), we conclude that the inequality (3.8) holds true for all $z \in \mathcal{U}$ and $t \geq 0$. Since all the conditions of Theorem 2.1 are satisfied, it follows that $L(z, t)$ is a Loewner chain and hence the function $L(z, 0) = F_\alpha(z)$ is analytic and univalent in \mathcal{U} . \square

Suitable choices of the functions g and h in Theorem 3.1 yield various univalence criteria as follows.

For $g(z) \equiv f'(z)$, $h(z) \equiv -\frac{1}{2\alpha} \frac{f''(z)}{f'(z)}$, we obtain the next result.

Corollary 3.1. *Let α be a complex number, $\Re\alpha > 0$, $m \in \mathbb{R}$, $m > 0$ and $f \in \mathcal{A}$. If, for all $z \in \mathcal{U} \setminus \{0\}$, the inequality*

$$\left| \frac{(1 - |z|^{\alpha(m+1)})^2}{2\alpha^2 |z|^{\alpha(m+1)}} \left(z^2 \{f; z\} + (1 - \alpha) \frac{zf''(z)}{f'(z)} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2}, \tag{3.13}$$

holds true, then the function F_α defined by (3.3) is analytic and univalent in \mathcal{U} , where $\{f; z\}$ denotes Schwarzian derivative of $f(z)$ which is defined by

$$\{f; z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)}\right)^2. \tag{3.14}$$

Example 3.1. Consider the function $f \in \mathcal{A}$ defined by

$$f'(z) = \frac{1}{(1 - z^\alpha)^2}, \quad z \in \mathcal{U}.$$

Then, some elementary calculations show that

$$\{f; z\} = \frac{2\alpha(\alpha - 1)z^{\alpha-2}}{1 - z^\alpha} \quad \text{and} \quad z^2 \{f; z\} + (1 - \alpha) \frac{zf''(z)}{f'(z)} = 0.$$

Therefore, the function $f(z)$ satisfies the condition (3.13). Furthermore, by using (3.3), we obtain that

$$F_\alpha(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u) du\right)^{1/\alpha} = \left(\alpha \int_0^z \frac{u^{\alpha-1}}{(1 - u^\alpha)^2} du\right)^{1/\alpha} = \frac{z}{(1 - z^\alpha)^{1/\alpha}}.$$

For this function $F_\alpha(z)$, we see that

$$\Re\left(\frac{zF'_\alpha(z)}{F_\alpha(z)}\right) = \Re\left(\frac{1}{1-z^\alpha}\right) > \frac{1}{2}, \quad z \in \mathcal{U},$$

which means that $F_\alpha(z)$ is starlike of order $1/2$ (univalent) in \mathcal{U} .

Remark 3.1. For special values of the parameters α and m in Corollary 3.1, we get some well known results:

- (i) For $m = 1$, we get the result given in the paper of Ovesea [15].
- (ii) For $\alpha = 1$, since $F_1(z) = f(z)$, Corollary 3.1 generalizes the criterion of univalence due to Nehari [13].
- (iii) For $\alpha = 1$ and $m = 1$, we obtain the univalence criterion due to Nehari [13].

Remark 3.2. For the function $f(z)$ considered in Example 3.1, we have that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) = \Re\left(1 - 2\alpha + \frac{2\alpha}{1-z^\alpha}\right) > 1 - \alpha, \quad z \in \mathcal{U}.$$

Thus, $f(z)$ is convex of order $1 - \alpha$ with $0 < \alpha \leq 1$. In this case, the integral operator $F_\alpha(z)$ of $f(z)$ satisfies

$$\Re\left(\frac{zF'_\alpha(z)}{F_\alpha(z)}\right) = \Re\left(\frac{1}{1-z^\alpha}\right) > \frac{1}{2}, \quad z \in \mathcal{U}$$

and

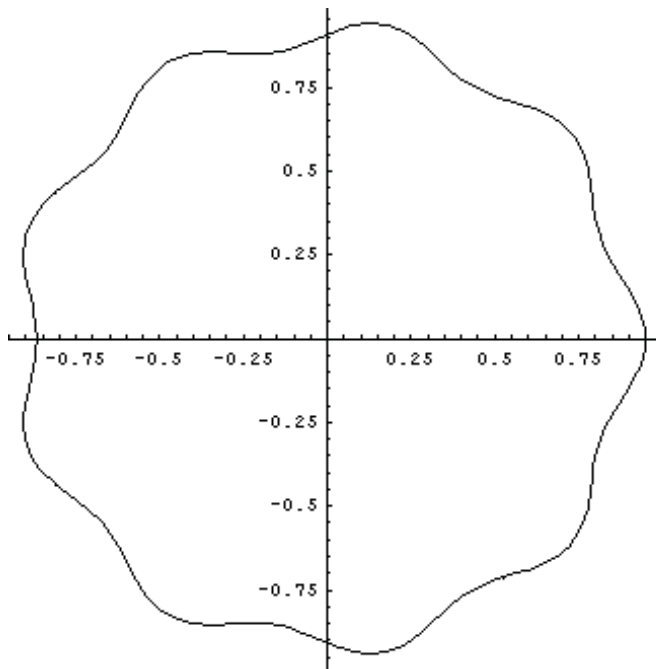
$$\Re\left(1 + \frac{zF''_\alpha(z)}{F'_\alpha(z)}\right) = \Re\left(\frac{1+\alpha}{1-z^\alpha} - \alpha\right) > \frac{1-\alpha}{2}, \quad z \in \mathcal{U}.$$

Therefore, we can say that $F_\alpha(z)$ is starlike of order $1/2$ and convex of order $(1 - \alpha)/2$ with $0 < \alpha \leq 1$.

If we take $\alpha = 9$ in Example 3.1, then the function

$$F_9(z) = \frac{z}{(1-z^9)^{\frac{1}{9}}}$$

maps the circle with the center at the origin and radius 0.9 to the following closed curve.



In view of Remark 3.2, we give the following conjecture.

Conjecture 3.1 *If $f \in \mathcal{A}$ satisfies*

$$z^2 \{f; z\} + (1 - \alpha) \frac{zf''(z)}{f'(z)} = 0, \quad z \in \mathcal{U},$$

then the integral operator $F_\alpha(z)$ defined by (3.3) is starlike of order $1/2$ and convex of order $(1 - \alpha)/2$ with $0 < \alpha \leq 1$.

For $g(z) \equiv f'(z)$ and $h(z) \equiv \frac{1}{\alpha} \left(\frac{1}{z} - \frac{f'(z)}{f(z)} \right)$, we have

Corollary 3.2. *Let α be a complex number, $\Re \alpha > 0$, $m \in \mathbb{R}$, $m > 0$ and $f \in \mathcal{A}$. If, for all $z \in \mathcal{U} \setminus \{0\}$, the inequality*

$$\left| \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \cdot z \frac{d}{dz} \log \left(\frac{z^2 f'(z)}{f^2(z)} \right) + \frac{(1 - |z|^{\alpha(m+1)})^2}{\alpha^2 |z|^{\alpha(m+1)}} \cdot z \frac{d}{dz} \left(\log \frac{z^{1+\alpha} f'(z)}{f^{1+\alpha}(z)} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.15}$$

holds true, then the function F_α defined by (3.3) is analytic and univalent in \mathcal{U} .

Remark 3.3. Corollary 3.2 is a generalization of the univalence criterion due to Goluzin [8].

- (i) For $m = 1$ in Corollary 3.2, we get the results from [8] and for $\alpha = 1$ the one from [23].
- (ii) The case $\alpha = 1$ and $m = 1$ gives Goluzin’s criterion [8].

For $g(z) \equiv \left(\frac{f(z)}{z} \right)^2$ and $h(z) \equiv \frac{1}{\alpha} \left(\frac{1}{z} - \frac{f(z)}{z^2} \right)$, we get

Corollary 3.3. Let α be a complex number, $\Re\alpha > 0$, $m \in \mathbb{R}$, $m > 0$ and $f \in \mathcal{A}$. If f satisfies the inequalities

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{3.16}$$

and

$$\begin{aligned} & \left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) |z|^{\alpha(m+1)} + 2 \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) \right. \\ & \left. \frac{(1 - |z|^{\alpha(m+1)})^2}{\alpha^2 |z|^{\alpha(m+1)}} \left[\left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) + (1 - \alpha) \left(\frac{f(z)}{z} - 1 \right) - \frac{m-1}{2} \right] \right| \leq \frac{m+1}{2} \end{aligned} \tag{3.17}$$

for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α defined by (3.3) is analytic and univalent in \mathcal{U} .

Remark 3.4. Corollary 3.3 represents a generalization of the univalence criterion due to Ozaki and Nunokawa [17].

- (i) For $m = 1$, we get the result from [24] and for $\alpha = 1$ the one given in [22].
- (ii) In the case $m = 1$ and $\alpha = 1$, Corollary 3.3 reduces to the univalence criterion obtained by Ozaki and Nunokawa [17].

For $g(z) \equiv f'(z)$, $h(z) \equiv 0$ we get

Corollary 3.4. Let α be a complex number, $\Re\alpha > 0$, $m \in \mathbb{R}$, $m > 0$ and $f \in \mathcal{A}$. If f satisfies

$$\left| \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \cdot \frac{z f''(z)}{f'(z)} - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.18}$$

for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α defined by (3.3) is analytic and univalent in \mathcal{U} .

Remark 3.5. Corollary 3.4 represents a generalization of the well known univalence criterion due to Becker [3].

- (i) For $m = 1$, we obtain the result from [18].
- (ii) In the case $m = 1$ and $\alpha = 1$, the above corollary reduces to Becker's criterion [3].

For $h(z) \equiv 0$, $g(z) \equiv \frac{f'(z)(p(z)+1)}{2}$, where p is analytic in \mathcal{U} , $p(0) = 1$, we have

Corollary 3.5. Let α be a complex number, $\Re\alpha > 0$, $m \in \mathbb{R}$, $m > 0$ and $f \in \mathcal{A}$. If there exists an analytic functions p in \mathcal{U} such that $p(0) = 1$ and the inequalities

$$\left| \frac{1 - p(z)}{1 + p(z)} - \frac{m-1}{2} \right| < \frac{m+1}{2} \tag{3.19}$$

and

$$\left| \frac{1 - p(z)}{1 + p(z)} |z|^{\alpha(m+1)} + \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \left(\frac{z f''(z)}{f'(z)} + \frac{z p'(z)}{p(z)+1} \right) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.20}$$

hold true for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α defined by (3.3) is analytic and univalent in \mathcal{U} .

For Corollary 3.5, we give the following example.

Example 3.2. Let α be a complex number such that $\Re\alpha > 0$ and $m \in \mathbb{R}$, $m > 1$. Then, we consider the functions

$$f(z) = z + \frac{e^{i\theta}}{2} z^2$$

and

$$p(z) = \frac{1 - e^{i\theta} z}{1 + e^{i\theta} z}.$$

We note that

$$\frac{zf''(z)}{f'(z)} = \frac{e^{i\theta} z}{1 + e^{i\theta} z},$$

and

$$\frac{1 - p(z)}{1 + p(z)} = e^{i\theta} z, \quad \frac{zp'(z)}{p(z) + 1} = -\frac{e^{i\theta} z}{1 + e^{i\theta} z}.$$

Therefore, we have that

$$\left| \frac{1 - p(z)}{1 + p(z)} - \frac{m-1}{2} \right| = \left| e^{i\theta} z - \frac{m-1}{2} \right| \leq |z| + \frac{m-1}{2} < 1 + \frac{m-1}{2} = \frac{m+1}{2}$$

and

$$\begin{aligned} & \left| \frac{1 - p(z)}{1 + p(z)} |z|^{\alpha(m+1)} + \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z) + 1} \right) - \frac{m-1}{2} \right| \\ & \leq |z|^{(m+1)\Re\alpha+1} + \frac{m-1}{2} < 1 + \frac{m-1}{2} = \frac{m+1}{2} \end{aligned}$$

for all $z \in \mathcal{U}$, $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$ and $m > 1$.

It follows that all the conditions of Corollary 3.5 are satisfied. By using (3.3), we obtain that

$$\begin{aligned} F_\alpha(z) &= \left(\alpha \int_0^z u^{\alpha-1} f'(u) du \right)^{1/\alpha} = \left(\alpha \int_0^z u^{\alpha-1} (1 + e^{i\theta} u) du \right)^{1/\alpha} \\ &= z \left(1 + \frac{\alpha}{\alpha+1} e^{i\theta} z \right)^{1/\alpha}. \end{aligned}$$

For this function $F_\alpha(z)$, we have

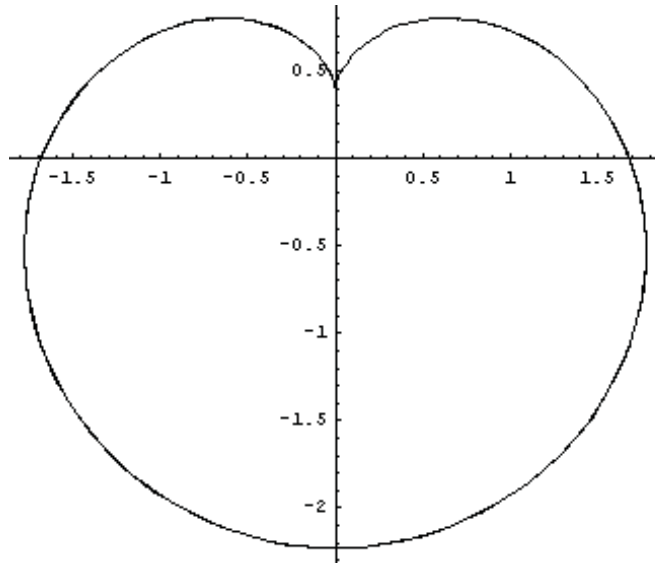
$$\Re \left(\frac{zF'_\alpha(z)}{F_\alpha(z)} \right) = \Re \left(\frac{1}{\alpha} \left(\alpha + 1 - \frac{1}{1 + \frac{\alpha}{\alpha+1} e^{i\theta} z} \right) \right) > 0$$

for $z \in \mathcal{U}$. Thus, $F_\alpha(z)$ is starlike (univalent) in \mathcal{U} .

Remark 3.6. If we take $\alpha = 1/6$ and $\theta = \pi/2$ in Example 3.2, then we have that

$$F_{1/6}(z) = z \left(1 + \frac{i}{7} z \right)^6.$$

This function $F_{1/6}(z)$ maps the unit circle to the following starlike curve.



In view of 3.6, we give the following conjecture.

Conjecture 3.2. *If $f \in \mathcal{A}$ satisfies*

$$\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1} = 0$$

for some analytic function $p(z)$ such that $p(0) = 1$ and

$$\left| \frac{1-p(z)}{1+p(z)} \right| < 1, \quad z \in \mathcal{U},$$

then $F_\alpha(z)$ defined by (3.3) is starlike in \mathcal{U} .

Remark 3.7. Corollary 3.5 represents a generalization of the univalence criterion due to Lewandowski [12].

- (i) For $m = 1$, we obtain the result from [14].
- (ii) In the case $m = 1$ and $\alpha = 1$, the above corollary reduces to Lewandowski's criterion [12].

Remark 3.8. Theorem 3.1 gives us a connection between the univalence criteria of Becker, Lewandowski, Nehari, Goluzin and also of Ozaki and Nunokawa and their generalizations.

4. Quasiconformal extension

Now, we discuss the quasiconformal extension for the integral operator $F_\alpha(z)$.

Theorem 4.1. *Let α be a complex number, $\Re\alpha > 0$, $m \in \mathbb{R}$, $m > 0$, $k \in [0, 1)$ and $f \in \mathcal{A}$. If there exist two analytic functions in \mathcal{U} , $g(z) = 1 + b_1z + \dots$ and $h(z) = c_0 + c_1z + \dots$ such that the inequalities*

$$\left| \left(\frac{f'(z)}{g(z)} - 1 \right) - \frac{m-1}{2} \right| \leq k \frac{m+1}{2} \tag{4.1}$$

and

$$\begin{aligned} & \left| \left(\frac{f'(z)}{g(z)} - 1 \right) |z|^{\alpha(m+1)} + \frac{1 - |z|^{\alpha(m+1)}}{\alpha} \left(2\alpha \frac{zf'(z)h(z)}{g(z)} + \frac{zg'(z)}{g(z)} \right) \right. \\ & \left. + \frac{(1 - |z|^{\alpha(m+1)})^2}{\alpha|z|^{\alpha(m+1)}} z^2 \left[\alpha \frac{f'(z)h^2(z)}{g(z)} + \frac{g'(z)h(z)}{g(z)} + (\alpha - 1) \frac{h(z)}{z} - h'(z) \right] - \frac{m-1}{2} \right| \leq k \frac{m+1}{2} \end{aligned} \tag{4.2}$$

hold true for all $z \in \mathcal{U} \setminus \{0\}$, then the function F_α given by (3.3) has a quasiconformal extension to \mathbb{C} .

Proof. In the proof of Theorem 3.1, we have shown that the function $L(z, t)$ given by (3.4) is a subordination chain in \mathcal{U} . Applying Theorem 2.2 to the function $w(z, t)$ given by (3.6), we obtain that the condition

$$\left| \frac{(1+a)\mathcal{G}(z, t) + 1 - ma}{(1-a)\mathcal{G}(z, t) + 1 + ma} \right| \leq l, \quad z \in \mathcal{U}, \quad t \geq 0 \text{ and } l \in [0, 1), \tag{4.3}$$

where $\mathcal{G}(z, t)$ is defined by (3.7), implies l -quasiconformal extensibility of F_α .

Lengthy but elementary calculation shows that the last inequality (4.3) is equivalent to

$$\left| \mathcal{G}(z, t) - \frac{a(1+l^2)(m-1) + (1-l^2)(ma^2-1)}{2a(1+l^2) + (1-l^2)(1+a^2)} \right| \leq \frac{2al(1+m)}{2a(1+l^2) + (1-l^2)(1+a^2)}. \tag{4.4}$$

It is easy to check that, under the assumption (4.2), we have

$$\left| \mathcal{G}(z, t) - \frac{m-1}{2} \right| \leq k \frac{m+1}{2}. \tag{4.5}$$

Consider the two disks Δ and Δ' defined by (4.4) and (4.5) respectively, where $\mathcal{G}(z, t)$ is replaced by a complex variable ζ . Our theorem will be proved if we find the smallest $l \in [0, 1)$ for which Δ' is contained in Δ . This will be so if and only if the distance apart of the centers plus the smallest radius is equal, at most, to the largest radius. So, we are required to prove that

$$\left| \frac{a(1+l^2)(m-1) + (1-l^2)(ma^2-1)}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{m-1}{2} \right| + k \frac{m+1}{2} \leq \frac{2al(1+m)}{2a(1+l^2) + (1-l^2)(1+a^2)}$$

or equivalently

$$\frac{(1-l^2)|1-a^2|}{2[2a(1+l^2) + (1-l^2)(1+a^2)]} \leq \frac{2al}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{k}{2} \tag{4.6}$$

with the condition

$$\frac{2al}{2a(1+l^2) + (1-l^2)(1+a^2)} - \frac{k}{2} \geq 0. \quad (4.7)$$

We will solve inequalities (4.6) and (4.7) for $1 - a^2 > 0$. In a similar way, they can be solved for $1 - a^2 < 0$.

It is easy to check that the solution of inequality (4.6) is $L_1 \leq l$ and $l \leq L_2$, where

$$L_1 = \frac{(1-a)^2 + k(1-a^2)}{1-a^2 + k(1-a)^2}, \quad L_2 = -\frac{(1+a)^2 + k(1-a^2)}{1-a^2 + k(1-a)^2}.$$

Since $L_2 < 0$, it remains $L_1 \leq l < 1$.

From the inequality (4.7), we obtain $\mathcal{L}_1 \leq l$ and $l \leq \mathcal{L}_2$, where

$$\mathcal{L}_1 = \frac{-2a + \sqrt{4a^2 + (1-a^2)^2 k^2}}{k(1-a)^2}, \quad \mathcal{L}_2 = \frac{-2a - \sqrt{4a^2 + (1-a^2)^2 k^2}}{k(1-a)^2}.$$

Since $\mathcal{L}_2 < 0$ and $\mathcal{L}_1 \leq L_1$, we get $L_1 \leq l < 1$.

If $a = 1$, (4.6) and (4.7) reduce to $k \leq l$ and thus $l = k$.

Consequently, the function F_α has an l -quasiconformal extension to \mathbb{C} , where

$$l = \frac{(1-a)^2 + k|1-a^2|}{|1-a^2| + k(1-a)^2} \text{ if } a \in (0, \infty) \setminus \{1\} \text{ and } l = k \text{ if } a = 1. \quad \square$$

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