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INTEGRAL VERSIONS OF SOME GRÜSS TYPE INEQUALITIES

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Abstract. In this paper we shall prove integral version of integral version of some Grüss type inequalities.

1. Introduction

Let functions f, $g : [a, b] \to R$ be integrable functions and let $p : [a, b] \to R$ be positive integrable function. Then Grüss type inequality is an estimation of difference:

$$I(a, b; p) := \int_a^b p(t)dt \int_a^b p(t)f(t)g(t)dt - \int_a^b p(t)f(t)dt \int_a^b p(t)g(t)dt. \tag{1}$$

Let functions f, $g : [a, b] \to R$ be integrable, both increasing or both decreasing and $p : [a, b] \to R$ be positive integrable function. Then

$$I(a, b; p) \geq 0. \tag{2}$$

This is well-known as Čebyšev's inequality [1, pp.239].

If functions f, $g : [a, b] \to R$ be integrable and $p : [a, b] \to R$. Further let

$$\varphi \leq f(x) \leq \Phi, \quad \gamma \leq g(x) \leq \Gamma. \tag{3}$$

Then

$$\frac{I(a, b; 1)}{(b-a)^2} \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma). \tag{4}$$

This is well-known as Grüss type inequality [1, pp.295].

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be n-tuples of real numbers and let $p = (p_1, \ldots, p_n)$ be an n-tuple of positive numbers. Then discrete Grüss type inequality is an estimation of difference:

$$I(a, b; p) := \sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i. \tag{5}$$

J. Pečarić [2] has proved:

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Theorem A. *Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be, both increasing or both decreasing, n-tuples of real numbers and let $p = (p_1, \dots, p_n)$ be an n-tuple of positive numbers. Then we have the inequality:*

$$I(a, b; p) \le |a_n - a_1| \cdot |b_n - b_1| \cdot \max_{1 \le k \le n} (P_k \overline{P}_k) \tag{6}$$

where $P_k = \sum_{i=1}^{k} p_i$, $\quad \overline{P}_k = P_n - P_k$.

D. Andrica and C. Badea [3] have proved:

Theorem B. *Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be n-tuple of real numbers and let $p = (p_1, \dots, p_n)$ be an n-tuple of positive numbers. Further let a, b*

$$m_1 \le a_i \le M_1, \quad m_2 \le a_i \le M_2. \tag{7}$$

Then the following inequality is valid

$$I(a, b; p) \le |M_1 - m_1| \cdot |M_2 - m_2| \cdot \max_{J \subset I_n} (P(J)(P_n - P(J))) \tag{8}$$

where $I_n = \{1, \dots, n\}$ *and* $P(J) = \sum_{i \in J} p_i$ *for* $J \subset I_n$.

S. S. Dragomir and R. P. Agarwal [4] prove:

Theorem C. *Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be n-tuples of complex numbers and let $p = (p_1, \dots, p_n)$ be probability distribution, i.e. $p_i \ge 0$ $(i = 1, \dots, n)$ and $\sum_{i=1}^{n} p_i = 1$. Then we have the following inequalities*

$$|I(a, b; p)| \le \left\{ \begin{array}{c} \max\limits_{k=1,\dots,n-1} |a_{k+1} - a_k| \max\limits_{i=1,\dots,n} |b_i| \sum\limits_{i,j=1}^{n} p_i p_j |i - j| \\ n^{1-\frac{1}{q}} \max\limits_{i=1,\dots,n-1} |a_{k+1} - a_k| \left(\sum\limits_{i=1}^{n} |b_i|^{\frac{q}{q-1}} \right)^{1-\frac{1}{q}} \left(\sum\limits_{i,j=1}^{n} p_i^q p_j^q |i - j|^q \right)^{\frac{1}{q}} (q > 1) \\ n \max\limits_{i=1,\dots,n-1} |a_{k+1} - a_k| \sum\limits_{i=1}^{n} |b_i| \max\limits_{i,j=1,\dots,n} \{p_i p_j |i - j|\} \end{array} \right\}$$
$$= F_1(a, b; p) \tag{9}$$

and

$$|I(a, b; p)| \le \left\{ \begin{array}{c} \max\limits_{k=1,\dots,n-1} |a_{k+1} - a_k| \max\limits_{i=1,\dots,n} \{p_i |b_i|\} \sum\limits_{i,j=1}^{n} p_i |i - j| \\ n^{1-\frac{1}{q}} \max\limits_{i=1,\dots,n-1} |a_{k+1} - a_k| \left(\sum\limits_{i=1}^{n} p_i |b_i|^{\frac{q}{q-1}} \right)^{1-\frac{1}{q}} \left(\sum\limits_{i,j=1}^{n} p_i p_j |i - j|^q \right)^{\frac{1}{q}} (q > 1) \\ (n-1) \max\limits_{i=1,\dots,n-1} |a_{k+1} - a_k| \sum\limits_{i=1}^{n} p_i |b_i| \end{array} \right\}$$
$$= F_2(a, b; p). \tag{10}$$

S. Izumino, J. Pečarić and B. Tepeš [5] have proved:

Theorem D. *Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be n-tuples of complex numbers and let $p = (p_1, \ldots, p_n)$ be probability distribution, i.e. $p_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n p_i = 1$. Then*

$$\begin{aligned} I(a,b;p) &\leq \sum_{i=1}^n \left(\sum_{j=1}^{i-1} P_j |\Delta a_j| + \sum_{j=1}^{n-1} \overline{P}_{j+1} |\Delta a_j| \right) p_i |b_i| \\ &\leq \sum_{i=1}^n \left(\sum_{i=1}^n |i-j| p_j \right)^{1-\frac{1}{q}} \left(\sum_{j=1}^{i-1} P_j |\Delta a_j|^q + \sum_{j=1}^{n-1} \overline{P}_{j+1} |\Delta a_j|^q \right)^{\frac{1}{q}} p_i |b_i| (q > 1) \\ &\leq \max_{j=1,\ldots,n-1} |\Delta a_j| \sum_{i=1}^n \left(\sum_{j=1}^n |i-j| p_j \right) p_i |b_i| \leq F_{1,2}(a,b;p). \end{aligned} \tag{11}$$

Then J. Pečarić and B. Tepeš [6] have proved the following improvement of the last inequality in (10):

Theorem E. *Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be n-tuples of complex numbers and let $p = (p_1, \ldots, p_n)$ be probability distribution, i.e. $p_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n p_i = 1$. Then ž*

$$I(a,b;p) \leq \max \left\{ \sum_{j=1}^n p_j (j-1), \sum_{j=1}^n p_j (n-j) \right\} \max_{k=1,\ldots,n-1} |a_{k+1} - a_k| \sum_{i=1}^n p_i |b_i|. \tag{12}$$

2. Main Result

Lemma 1. *Let functions f, $g : [a,b] \to R$ be differentiable functions and let $p : [a,b] \to R$ be positive integrable function. Then*

$$I(a,b;p) = \int_a^b \left(\int_a^x P(t) df(t) - \int_x^b \overline{P}(t) df(t) \right) p(x) g(x) dx, \tag{13}$$

where $P(x) = \int_a^x p(t)dt$ and $\overline{P}(x) = \int_x^b p(t)dt = P(b) - P(x)$.

Proof. It is obviously that:

$$I(a,b;p) = \int_a^b p(x) g(x) \left(f(x) \int_a^b p(t) dt - \int_a^b p(t) f(t) dt \right) dx. \tag{14}$$

We have

$$\begin{aligned}\int_a^b p(x)f(x)dx &= f(b)\int_a^b p(t)dt - \int_a^b P(t)df(t)\\ &= f(b)\int_a^b p(t)dt - \int_a^b P(t)df(t) + f(x)\int_a^b p(t)dt - f(x)\int_a^b p(t)dt\\ &= f(x)P(b) + \int_x^b P(b)df(t) - \int_a^x P(t)df(t) - \int_x^b P(t)df(t)\\ &= f(x)P(b) - \int_a^x P(t)df(t) + \int_x^b \overline{P}(t)df(t). \qquad (15)\end{aligned}$$

Using (15) in (14) we prove Lemma 1.

Theorem 1. *Let functions $f,\ g : [a,b] \to R$ be differentiable functions and let $p : [a,b] \to R$ be positive integrable function. Then*

$$\begin{aligned}|I(f,g;p)| &\le \int_a^b \left(\int_a^x P(t)|f'(t)|dt + \int_x^b \overline{P}(t)|f'(t)|dt\right) p(x)|g(x)|dx\\ &\le \int_a^b \left(\int_a^b |x-t|p(t)dt\right)^{1-\frac{1}{q}} \left(\int_a^x P(t)|f'(t)|^q dt + \int_x^b \overline{P}(t)|f'(t)|^q dt\right)^{\frac{1}{q}} p(x)|g(x)|dx\\ &\le \max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t)p(t)dt\right) p(x)|g(x)|dx \le F_{1,2}(f,g;p), \qquad (16)\end{aligned}$$

where

$$F_1(f,g;p)$$

$$= \begin{cases} \max\limits_{x\in[a,b]} |f'(x)| \max\limits_{x\in[a,b]} |g(x)| \int_a^b \int_a^b p(t)p(x)|x-t|dtdx \\ (b-a)^{1-\frac{1}{q}} \max\limits_{x\in[a,b]} |f'(x)| \left(\int_a^b |g(x)|^{\frac{q}{q-1}} dx\right)^{1-\frac{1}{q}} \left(\int_a^b \int_a^b (p(t))^q (p(x))^q |x-t|^q dtdx\right)^{\frac{1}{q}} \quad (q>1) \\ (b-a) \max\limits_{x\in[a,b]} |f'(x)| \int_a^b |g(x)|dx \max\limits_{t,x\in[a,b]} \{p(t)p(x)|x-t|\} \end{cases} \qquad (17)$$

and

$$F_2(f,g;p) = \begin{cases} \max\limits_{x\in[a,b]}|f'(x)| \max\limits_{x\in[a,b]}\{p(x)|g(x)|\} \int_a^b \int_a^b p(t)|x-t|dtdx \\ \max\limits_{x\in[a,b]}|f'(x)| \left(\int_a^b p(t)dt\right)^{1-\frac{1}{q}} \left(\int_a^b p(x)|p(x)|^{\frac{q}{q-1}}dx\right)^{1-\frac{1}{q}} \left(\int_a^b \int_a^b p(t)p(x)|x-t|^q dtdx\right)^{\frac{1}{q}} \\ \hfill (q>1) \\ \max\limits_{x\in[a,b]}|f'(x)| \max\left\{\int_a^b (b-x)p(x)dx, \int_a^b (x-a)p(x)dx\right\} \int_a^b p(x)|g(x)|dx. \end{cases} \tag{18}$$

Proof. Using Lemma 1 and $df = f'(x)dx$ we have

$$|I(f,g;p)| \leq \int_a^b \left(\int_a^x P(t)|f'(t)|dt + \int_x^b \overline{P}(t)|f'(t)|dt\right) p(x)|g(x)|dx. \tag{19}$$

Using fundamental inequality for means [1, pp.15] with $q > 1$ and (19) we have

$$\begin{aligned} &\int_a^b \left(\int_a^x P(t)|f'(t)|dt + \int_x^b \overline{P}(t)|f'(t)|dt\right) p(x)|g(x)|dx \\ &= \int_a^b \left(\int_a^x P(t)dt + \int_x^b \overline{P}(t)dt\right) \frac{(\int_a^x P(t)|f'(t)|dt + \int_x^b \overline{P}(t)|f'(t)|dt)}{\int_a^x P(t)dt + \int_x^b \overline{P}(t)dt} p(x)|g(x)|dx \\ &\leq \int_a^b \left(\int_a^x P(t)dt + \int_x^b \overline{P}(t)dt\right) \frac{(\int_a^x P(t)|f'(t)|^q dt + \int_x^b \overline{P}(t)|f'(t)|^q dt)^{\frac{1}{q}}}{(\int_a^x P(t)dt + \int_x^b \overline{P}(t)dt)^{\frac{1}{q}}} p(x)|g(x)|dx \\ &= \int_a^b \left(\int_a^x P(t)dt + \int_x^b \overline{P}(t)dt\right)^{1-\frac{1}{q}} \left(\int_a^x P(t)|f'(t)|^q dt + \int_x^b \overline{P}(t)|f'(t)|^q dt\right)^{\frac{1}{q}} p(x)|g(x)|dx. \end{aligned} \tag{20}$$

It is obvious that:

$$\int_a^x P(t)dt = \int_a^x \int_a^t p(s)dsdt = \int_a^x \int_s^x p(s)dtds = \int_a^x (x-s)p(s)ds = \int_a^x (x-t)p(t)dt, \tag{21}$$

and

$$\int_x^b \overline{P}(t)dt = \int_x^b \int_t^b p(s)dsdt = \int_x^b \int_x^s p(s)dtds = \int_x^b (s-x)p(s)ds = \int_x^b (t-x)p(t)dt. \tag{22}$$

Using (21), (22) and (20) we have second inequality in (16).

Fundamental inequality for means [1, pp.15] with $q=\infty$ second inequality in (16) we have the third inequality in (16):

$$\int_a^b \left(\int_a^b |x-t|p(t)dt\right)^{1-\frac{1}{q}} \left(\int_a^x P(t)|f'(t)|^q dt + \int_x^b \overline{P}(t)|f'(t)|^q dt\right)^{\frac{1}{q}} p(x)|g(x)|dx$$

$$\leq \max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t|p(t)dt\right) p(x)|g(x)dx. \tag{23}$$

It is obviously that

$$\max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t|p(t)dt\right) p(x)|g(x)|dx$$
$$\leq \max_{x\in[a,b]} |f'(x)| \max_{x\in[a,b]} |g(x)| \int_a^b \left(\int_a^b |x-t|p(t)dt\right) p(x)dx. \tag{24}$$

Using integral analogue of Hölder's inequality [1, pp.106] with $q>1$, we have

$$\max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t|p(t)dt\right) p(x)|g(x)|dx$$
$$\leq \max_{x\in[a,b]} |f'(x)| \left(\int_a^b \int_a^b |g(x)|^{\frac{q}{q-1}} dxdt\right)^{1-\frac{1}{q}} \left(\int_a^b \int_a^b |x-t|^q (p(t))^q (p(x))^q dtdx\right)^{\frac{1}{q}} \leq$$
$$= (b-a)^{1-\frac{1}{q}} \max_{x\in[a,b]} |f'(x)| \left(\int_a^b |g(x)|^{\frac{q}{q-1}} dx\right)^{1-\frac{1}{q}} \left(\int_a^b \int_a^b |x-t|^q (p(t))^q (p(x))^q dtdx\right)^{\frac{1}{q}}. \tag{25}$$

It is obviously that:

$$\max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t|p(t)dt\right) p(x)|g(x)|dx$$
$$\leq \max_{x\in[a,b]} |f'(x)| \max_{t,x\in[a,b]} \{p(t)p(x)|x-t|\} \int_a^b \int_a^b |g(x)|dtdx$$
$$= (b-a) \max_{x\in[a,b]} |f'(x)| \max_{t,x\in[a,b]} \{p(t)p(x)|x-t|\} \int_a^b |g(x)|dx. \tag{26}$$

It is also obviously that:

$$\max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t|p(t)dt\right) p(x)|g(x)|dx$$
$$\leq \max_{x\in[a,b]} |f'(x)| \max_{x\in[a,b]} \{p(x)|g(x)|\} \int_a^b \left(\int_a^b |x-t|p(t)dt\right) dx. \tag{27}$$

Using integral analogue of Hölder's inequality [1, pp.106] with $q > 1$, we have

$$\begin{aligned}
&\max_{x\in[a,b]} |f'(x)| \int_a^b \int_a^b |x-t| \cdot |g(x)| p(t)p(x)dtdx \\
&= \max_{x\in[a,b]} |f'(x)| \int_a^b \int_a^b |g(x)|(p(t)p(x))^{\frac{q-1}{q}} |x-t| (p(t)p(x))^{\frac{1}{q}} dtdx \\
&\le \max_{x\in[a,b]} |f'(x)| \left(\int_a^b \int_a^b |g(x)|^{\frac{q}{q-1}} p(t)p(x)dtdx \right)^{1-\frac{1}{q}} \left(\int_a^b \int_a^b |x-t|^q p(t)p(x)dtdx \right)^{\frac{1}{q}} \\
&= \max_{x\in[a,b]} |f'(x)| \left(\int_a^b p(t)dt \right)^{1-\frac{1}{q}} \left(\int_a^b |g(x)|^{\frac{q}{q-1}} p(x)dx \right)^{1-\frac{1}{q}} \left(\int_a^b \int_a^b |x-t|^q p(t)p(x)dtdx \right)^{\frac{1}{q}}.
\end{aligned} \tag{28}$$

It is obviously that

$$\begin{aligned}
&\max_{x\in[a,b]} |f'(x)| \int_a^b \left(\int_a^b |x-t| p(t)dt \right) p(x)|g(x)|dx \\
&\le \max_{x\in[a,b]} |f'(x)| \max_{x\in[a,b]} \left\{ \int_a^b |x-t|p(t)dt \right\} \int_a^b p(x)|g(x)|dx \\
&= \max_{x\in[a,b]} |f'(x)| \max \left\{ \int_a^b (b-x)p(x)dx, \int_a^b (x-a)p(x)dx \right\} \int_a^b p(x)|g(x)|dx.
\end{aligned} \tag{29}$$

Last equality in (29) is because function $y : [a,b] \to R$, $y(x) = \int_a^b |x-t|p(t)dt$ is convex function on $[a,b]$ or $y'(x) = P(x) - \overline{P}(x)$ and $y''(x) = 2p(x) > 0$.

Remark. Let $p : [a,b] \to R$ be positive integrable function. Then

$$\max \left\{ \int_a^b (b-x)p(x)dx, \int_a^b (x-a)p(x)dx \right\} \le (b-a) \int_a^b p(x)dx. \tag{30}$$

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