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INEQUALITIES SIMILAR TO OPIAL'S INEQUALITY INVOLVING HIGHER ORDER DERIVATIVES

B. G. PACHPATTE

Abstract. In this paper we establish some new inequalities similar to Opial's inequality involving functions and their higher order derivatives. The analysis used in the proofs is elementary and our results provide a new range of inequalities of this type.

1. Introduction

In 1960, Z. Opial [4] established the following interesting inequality.
If $y(t) > 0$ in (a, b), $y(a) = y(b) = 0$ and $y \in C^1[a, b]$, then

$$\int_a^b |y(t)y'(t)|dt \leq \frac{1}{4}(b-a)\int_a^b |y'(t)|^2 dt, \tag{1}$$

where the constant $\frac{1}{4}(b-a)$ is the best possible.

After the discovery of the inequality (1) by Opial [4] a large number of papers dealing with successively simpler proofs, numerous variants, generalizations and extensions of the inequality (1) have appeared in the literature, see [1-3, 5, 6] and the references given therein. The main object of this paper is to establish some new inequalities similar to Opial's inequality involving functions and their higher order derivatives. An interesting feature of the inequalities established here is that the analysis used in their proofs is elementary and our results provide new estimates on inequalities of this type.

2. Statement of Results

In order to formulate our results conveniently, we set the following notations.

$$M_1 = \frac{1}{\sqrt{3}(n-m-2)!}\left\{\int_a^b (t-a)\left(\int_a^t (t-s)^{n-m-2}(s-a)^{\frac{1}{2}}\frac{1}{r(s)}ds\right)^2 dt\right\}^{\frac{1}{2}}, \tag{2}$$

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$$M_2 = \frac{1}{\sqrt{3}\sqrt{2(n-2)+1}\sqrt{2(n-m-2)+1}(n-2)!(n-m-2)!}$$
$$\times \left\{ \int_a^b (t-a)^{2(n-m-2)+1} \left(\int_a^t \frac{1}{r(s)} (s-a)^{\frac{2(n-2)+1}{2}} ds \right)^2 dt \right\}^{\frac{1}{2}}, \tag{3}$$

$$M_3 = \frac{1}{\sqrt{3}\sqrt{2n-1}\sqrt{2n-2m-1}(n-1)!(n-k-1)!(n-m-1)!}$$
$$\times \left\{ \int_a^b (t-a)^{2n-2m-1} \left(\int_a^t (t-s)^{n-k-1} (s-a)^{\frac{2n-1}{2}} \frac{1}{r(s)} ds \right)^2 dt \right\}^{\frac{1}{2}}, \tag{4}$$

where $r(t)$ is some suitable function defined on $I = [a, b]$; $a < b$ are real constants. Throught, we assume that the integrals on the right sides of (2), (3), (4) are finite.

Our main results are given in the following theorems.

Theorem 1. *Let $n \geq 2$, $0 \leq m \leq n-2$ be integers. Let $r(t) > 0$ be of class C^1 on I and $y(t)$ be of class C^n on I satisfying $y(a) = 0$, $y^{(i-1)}(a) = 0$, $i = 2, 3, \ldots, n$ and $\int_a^b |(r(t)y^{(n-1)}(t))'|^2 dt < \infty$. Then*

$$\int_a^b |y^{(m)}(t)|\, |r(t)y^{(n-1)}(t)|\, |(r(t)y^{(n-1)}(t))'| dt \leq M_1 \left(\int_a^b |(r(t)y^{(n-1)}(t))'|^2 dt \right)^{\frac{3}{2}}, \tag{5}$$

where M_1 is defined by (2).

Theorem 2. *Let $n \geq 2$, $0 \leq m \leq n-2$ be integers. Let $r(t) > 0$ be of class C^{n-1} on I and $y(t)$ be of class C^n on I satisfying $y(a) = 0$, $(r(a)y'(a))^{(i-2)} = 0$, $i = 2, 3, \ldots, n$ and $\int_a^b |(r(t)y'(t))^{(n-1)}|^2 dt < \infty$. Then*

$$\int_a^b |y(t)|\, |(r(t)y'(t))^{(m)}|\, |(r(t)y'(t))^{(n-1)}| dt \leq M_2 \left(\int_a^b |(r(t)y'(t))^{(n-1)}|^2 dt \right)^{\frac{3}{2}}, \tag{6}$$

where M_2 is defined in (3).

Theorem 3. *Let $n \geq 1$, $0 \leq k \leq n-1$, $0 \leq m \leq n-1$ be integers. Let $r(t) > 0$ be of class C^n on I and $y(t)$ be of class C^{2n} on I satisfying $y^{(i-1)}(a) = 0$, $(r(a)y^{(n)}(a))^{(i-1)} = 0$ for $i = 1, 2, \ldots, n$ and $\int_a^b |(r(t)y^{(n)}(t))^{(n)}|^2 dt < \infty$. Then*

$$\int_a^b |y^{(k)}(t)|\, |(r(t)y^{(n)}(t))^{(m)}|\, |(r(t)y^{(n)}(t))^{(n)}| dt \leq M_3 \left(\int_a^b |(r(t)y^{(n)}(t))^{(n)}|^2 dt \right)^{\frac{3}{2}}, \tag{7}$$

where M_3 is defined in (4).

Remark 1. We note that in the various special cases the inequalities in Theorems 1-3 reduces to the new inequalities of the Opial type involving functions and their derivatives.

In particular, by taking $m = 0$, $n = 2$ and $r(t) = 1$, the inequality (5) in Theorem 1 reduces to the following inequality

$$\int_a^b |y(t)|\,|y'(t)|\,|y''(t)|dt \le \overline{M}_1 \left(\int_a^b |y''(t)|^2 dt\right)^{\frac{3}{2}},$$

where $\overline{M}_1$ denotes the constant obtained from the right side of (2) by taking $m = 0$, $n = 2$ and $r(t) = 1$.

3. Proof of Theorem 1

From the hypotheses and Taylor expansion we have

$$r(t)y^{(n-1)}(t) = \int_a^t (r(s)y^{(n-1)}(s))'ds, \tag{8}$$

$$y^{(m)}(t) = \frac{1}{(n-m-2)!}\int_a^t (t-s)^{n-m-2}y^{(n-1)}(s)ds. \tag{9}$$

Using (8) in (9) we have

$$y^{(m)}(t) = \frac{1}{(n-m-2)!}\int_a^t (t-s)^{n-m-2}\frac{1}{r(s)}\int_a^s (r(\sigma)y^{(n-1)}(\sigma))'d\sigma ds. \tag{10}$$

From (8), (10) and using Schwarz ineuqality we observe that

$$\begin{aligned} |r(t)y^{(n-1)}(t)| &\le \int_a^t |(r(s)y^{(n-1)}(s))'|ds \\ &\le (t-a)^{\frac{1}{2}}\left(\int_a^t |(r(s)y^{(n-1)}(s))'|^2 ds\right)^{\frac{1}{2}}. \end{aligned} \tag{11}$$

and

$$\begin{aligned} |y^{(m)}(t)| &\le \frac{1}{(n-m-2)!}\int_a^t (t-s)^{n-m-2}\frac{1}{r(s)}\int_a^s \left|\left(r(\sigma)y^{(n-1)}(\sigma)\right)'\right|d\sigma ds \\ &\le \frac{1}{(n-m-2)!}\int_a^t (t-s)^{n-m-2}\frac{1}{r(s)}(s-a)^{\frac{1}{2}}\left(\int_a^s \left|\left(r(\sigma)y^{(n-1)}(\sigma)\right)'\right|^2 d\sigma\right)^{\frac{1}{2}} ds \\ &\le \frac{1}{(n-m-2)!}\left(\int_a^t (t-s)^{n-m-2}\frac{1}{r(s)}(s-a)^{\frac{1}{2}}ds\right)\left(\int_a^t \left|\left(r(\sigma)y^{(n-1)}(\sigma)\right)'\right|^2 d\sigma\right)^{\frac{1}{2}}. \end{aligned} \tag{12}$$

From (11), (12) and using Schwarz inequality we observe that

$$\begin{aligned} &\int_a^b |y^{(m)}(t)|\,|r(t)y^{(n-1)}(t)|\,|(r(t)y^{(n-1)}(t))'|dt \\ &\le \frac{1}{(n-m-2)!}\int_a^b \left[(t-a)^{\frac{1}{2}}\left(\int_a^t (t-s)^{n-m-2}(s-a)^{\frac{1}{2}}\frac{1}{r(s)}ds\right)\right] \end{aligned}$$

$$\times\left[\left(\int_a^t |(r(s)y^{(n-1)}(s))'|^2 ds\right) |(r(t)y^{(n-1)}(t))'|\right] dt$$

$$\leq \frac{1}{(n-m-2)!}\left\{\int_a^b (t-a)\left(\int_a^t (t-s)^{n-m-2}(s-a)^{\frac{1}{2}}\frac{1}{r(s)}ds\right)^2 dt\right\}^{\frac{1}{2}}$$

$$\times\left\{\int_a^b \left(\int_a^t |(r(s)y^{(n-1)}(s))'|^2 ds\right)^2 |(r(t)y^{(n-1)}(t))'|^2 dt\right\}^{\frac{1}{2}}$$

$$= M_1\left(\int_a^b |(r(t)y^{(n-1)}(t))'|^2 dt\right)^{\frac{3}{2}}. \tag{13}$$

This completes the proof.

4. Proof of Theorem 2

From the hypotheses and Taylor expansion we have

$$r(t)y'(t) = \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2}(r(s)y'(s))^{(n-1)} ds, \tag{14}$$

$$(r(t)y'(t))^{(m)} = \frac{1}{(n-m-2)!}\int_a^b (t-s)^{n-m-2}(r(s)y'(s))^{(n-1)} ds. \tag{15}$$

From (14) we observe that

$$y(t) = \frac{1}{(n-2)!}\int_a^t \frac{1}{r(s)}\int_a^s (s-\sigma)^{n-2}(r(\sigma)y'(\sigma))^{(n-1)} d\sigma ds. \tag{16}$$

From (15) and (16) and using Schwarz inequality we observe that

$$|(r(t)y'(t))^{(m)}| \leq \frac{1}{(n-m-2)!}\int_a^t (t-s)^{n-m-2}|(r(s)y'(s))^{(n-1)}| ds$$

$$\leq \frac{1}{(n-m-2)!}\left(\int_a^t (t-s)^{2(n-m-2)} ds\right)^{\frac{1}{2}}\left(\int_a^t |(r(s)y'(s))^{(n-1)}|^2 ds\right)^{\frac{1}{2}}$$

$$= \frac{1}{(n-m-2)!}\frac{(t-a)^{\frac{2(n-m-2)+1}{2}}}{\sqrt{2(n-m-2)+1}}\left(\int_a^t |(r(s)y'(s))^{(n-1)}|^2 ds\right)^{\frac{1}{2}}, \tag{17}$$

and

$$|y(t)| \leq \frac{1}{(n-2)!}\int_a^t \frac{1}{r(s)}\left(\int_a^s (s-\sigma)^{n-2}|(r(\sigma)y'(\sigma))^{(n-1)}| d\sigma\right) ds$$

$$\leq \frac{1}{(n-2)!}\int_a^t \frac{1}{r(s)}\left(\int_a^s (s-\sigma)^{2(n-2)} d\sigma\right)^{\frac{1}{2}}\left(\int_a^s |(r(\sigma)y'(\sigma))^{(n-1)}|^2 d\sigma\right)^{\frac{1}{2}} ds$$

$$= \frac{1}{(n-2)!}\frac{1}{\sqrt{2(n-2)+1}}\left(\int_a^t \frac{1}{r(s)}(s-a)^{\frac{2(n-2)+1}{2}} ds\right)$$

$$\times \left(\int_a^t |(r(\sigma)y'(\sigma))^{(n-1)}|^2 d\sigma \right)^{\frac{1}{2}}. \tag{18}$$

From (17), (18) and using Schwarz inequality we observe that

$$\begin{aligned}
&\int_a^b |y(t)|\,|(r(t)y'(t))^{(m)}|\,|(r(t)y'(t))^{(n-1)}|dt \\
&\le \frac{1}{(n-2)!(n-m-2)!\sqrt{2(n-2)+1}\sqrt{2(n-m-2)+1}} \\
&\quad \times \int_a^b \left[(t-a)^{\frac{2(n-m-2)+1}{2}} \int_a^t \frac{1}{r(s)}(s-a)^{\frac{2(n-2)+1}{2}} ds \right] \\
&\quad \times \left[\left(\int_a^t |(r(s)y'(s))^{(n-1)}|^2 ds \right) |(r(t)y'(t))^{(n-1)}| \right] dt \\
&\le \frac{1}{(n-2)!(n-m-2)!\sqrt{2(n-2)+1}\sqrt{2(n-m-2)+1}} \\
&\quad \times \left\{ \int_a^b (t-a)^{2(n-m-2)+1} \left(\int_a^t \frac{1}{r(s)}(s-a)^{\frac{2(n-2)+1}{2}} ds \right)^2 dt \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \int_a^b \left(\int_a^t |(r(s)y'(s))^{(n-1)}|^2 ds \right)^2 |(r(t)y'(t))^{(n-1)}|^2 dt \right\}^{\frac{1}{2}} \\
&= M_2 \left(\int_a^b |(r(t)y'(t))^{(n-1)}|^2 dt \right)^{\frac{3}{2}}.
\end{aligned} \tag{19}$$

The proof is complete.

5. Proof of Theorem 3

From the hypotheses and Taylor expansion we have

$$y^{(k)}(t) = \frac{1}{(n-k-1)!} \int_a^t (t-s)^{n-k-1} y^{(n)}(s) ds, \tag{20}$$

$$r(t)y^{(n)}(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} (r(s)y^{(n)}(s))^{(n)} ds, \tag{21}$$

$$(r(t)y^{(n)}(t))^{(m)} = \frac{1}{(n-m-1)!} \int_a^t (t-s)^{n-m-1} (r(s)y^{(n)}(s))^{(n)} ds. \tag{22}$$

Using (21) in (20) we have

$$y^{(k)}(t) = \frac{1}{(n-1)!(n-k-1)!} \int_a^t (t-s)^{n-k-1} \frac{1}{r(s)} \int_a^s (s-\sigma)^{n-1} (r(\sigma)y^{(n)}(\sigma))^{(n)} d\sigma ds. \tag{23}$$

From (22) and (23) and using Schwarz inequality we observe that

$$\begin{aligned}
|(r(t)y^{(n)}(t))^{(m)}| &\leq \frac{1}{(n-m-1)!}\left(\int_a^t (t-s)^{2(n-m-1)}ds\right)^{\frac{1}{2}}\left(\int_a^t |(r(s)y^{(n)}(s))^{(n)}|^2 ds\right)^{\frac{1}{2}} \\
&= \frac{1}{(n-m-1)!\sqrt{2(n-m-1)+1}}(t-a)^{\frac{2(n-m-1)+1}{2}} \\
&\quad \times \left(\int_a^t |(r(s)y^{(n)}(s))^{(n)}|^2 ds\right)^{\frac{1}{2}}, \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
|y^{(k)}(t))| &\leq \frac{1}{(n-1)!(n-k-1)!}\int_a^t (t-s)^{n-k-1}\frac{1}{r(s)}\left(\int_a^s (s-\sigma)^{2(n-1)}d\sigma\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_a^s |(r(\sigma)y^{(n)}(\sigma))^{(n)}|^2 d\sigma\right)^{\frac{1}{2}} ds \\
&\leq \frac{1}{\sqrt{(2n-1)}(n-1)!(n-k-1)!}\left(\int_a^t (t-s)^{n-k-1}(s-a)^{\frac{2n-1}{2}}\frac{1}{r(s)}ds\right) \\
&\quad \times \left(\int_a^t |(r(\sigma)y^{(n)}(\sigma))^{(n)}|^2 d\sigma\right)^{\frac{1}{2}}. \qquad (25)
\end{aligned}$$

From (24), (25) and using Schwarz inequality we observe that

$$\begin{aligned}
&\int_a^b |y^{(k)}(t)|\,|(r(t)y^{(n)}(t))^{(m)}|\,|(r(t)y^{(n)}(t))^{(n)}|dt \\
&\leq \frac{1}{\sqrt{2n-1}\sqrt{2(n-m-1)+1}(n-1)!(n-k-1)!(n-m-1)!} \\
&\quad \times \int_a^b \left[\left(\int_a^t (t-s)^{n-k-1}(s-a)^{\frac{2n-1}{2}}\frac{1}{r(s)}ds\right)(t-a)^{\frac{2(n-m-1)+1}{2}}\right] \\
&\quad \times \left[\left(\int_a^t |(r(s)y^{(n)}(s))^{(n)}|^2 ds\right)|(r(t)y^{(n)}(t))^{(n)}|\right] dt \\
&\leq \frac{1}{\sqrt{2n-1}\sqrt{2n-2m+1}(n-1)!(n-k-1)!(n-m-1)!} \\
&\quad \times \left\{\int_a^b (t-a)^{2n-2m-1}\left(\int_a^t (t-s)^{n-k-1}(s-a)^{\frac{2n-1}{2}}\frac{1}{r(s)}ds\right)^2 dt\right\}^{\frac{1}{2}} \\
&\quad \times \left\{\int_a^b \left(\int_a^t |(r(s)y^{(n)}(s))^{(n)}|^2 ds\right)^2 |(r(t)y^{(n)}(t))^{(n)}|^2 dt\right\}^{\frac{1}{2}} \\
&= M_3\left(\int_a^b |(r(t)y^{(n)}(t))^{(n)}|^2 dt\right)^{\frac{3}{2}}. \qquad (26)
\end{aligned}$$

The proof is complete.

Remark 2. We note that one can very easily obtain the weighted versions of Theorems 1-3 by following the similar arguments as in [6]. We also note that, usually it is difficult to calculate the best possible constants in the general inequalities like those of given in Theorem 1-3. Here, it is to be noted that the constants obtained in Theorems 1-3 are probably not sharp, because in the general cases of Theorems 1-3, the steps in the proofs at which significant wastage may be occurring are (11), (12), (13), (17), (18), (19), (24), (25) and (26).

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57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431 001, (Maharashtra), India.

E-mail: bgpachpatte@hotmail.com