TAMKANG JOURNAL OF MATHEMATICS
Volume 44, Number 4, 431-444, Winter 2013
doi:10.5556/j.tkjm.44.2013.1232
Available online at http://journals.math.tku.edu.tw/

CO-SCREEN CONFORMAL HALF LIGHTLIKE SUBMANIFOLDS

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Abstract. In this paper, we introduce and study the geometry of half lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ satisfying that the shape operator of screen transversal bundle is conformal to the shape operator of lightlike transversal bundle of M. Using this geometric condition we obtain some results to characterize the unique existence of screen distribution of M, also, we present some sufficient conditions for the induced Ricci curvature tensor of M to be symmetric.

1. Introduction

It is well known that the intersection of the normal bundle and the tangent bundle of a submanifold of a semi-Riemannian manifold may be not trivial, it is more difficult and interesting to study the geometry of lightlike submanifolds than non-degenerate submanifolds. The two standard methods to deal with the above difficulties were developed by Kupeli [11] and Duggal-Bejancu [3, 7] respectively. However, unlike the Riemannian case the geometry of lightlike submanifolds depends on the choice of the screen distribution. So it is important to look for natural geometric condition to make the screen distribution exists uniquely. Also, the induced Ricci curvature tensor of a lightlike submanifold from ambient space may be not symmetric. If the Ricci curvature tensor is not symmetric, then it has no physical and geometric meaning and the scalar curvature of lightlike submanifold has no way to study.

It is obvious to see that there are two cases of codimension 2 lightlike submanifolds M of a semi-Riemannian manifold, since for this type the dimension of their radical distribution is either 1 or 2. A codimension 2 lightlike submanifold is called half lightlike submanifold [2] if dim(Rad(TM))=1. For more results about half lightlike submanifolds, we refer the reader to [8, 9, 10].

Duggal [2] obtained a theorem to characterize the unique existence of screen distribution of half lightlike submanifold by a reasonable geometric condition, that is, the subbundle F of

Received August 13, 2012, accepted December 11, 2012.
Communicated by Cherng-Yih Tu.
2010 *Mathematics Subject Classification*. 53C25, 53C50.
Key words and phrases. Half lightlike submanifold, semi-Riemannian manifold, co-screen conformal, induced Ricci tensor.
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$D^{\perp}$ admits a covariant constant non-null vector field and the first derivative $\mathscr{S}$ coincides with screen distribution. Also, by the definition of screen conformal half lightlike submanifold [2, 7] the other theorems to characterize the unique existence of screen distribution are obtained. Using the screen conformal condition, some geometric objects like the induced symmetric Ricci curvature tensor [2, 6, 7] and Einstein half lightlike submanifold [8] were investigated.

In this paper, we mainly discuss the properties of half lightlike submanifolds of semi-Riemannian manifolds with a condition that the shape operator of screen transversal distribution and the shape operator of lightlike transversal distribution are conformal. By using this reasonable geometric condition we obtain some new theorems to characterize the unique existence of screen distribution and induced symmetric Ricci curvature tensor.

2. Preliminaries

In this section, we follow [2, 7] due to K.L. Duggal and D.H. Jin for the notations and fundamental equations on half lightlike submanifolds of semi-Riemannian manifolds.

A submanifold (M, g) of dimension m immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of dimension $(m+n)$ is called a lightlike submanifold if the metric g induced from ambient space is degenerate and radical distribution $Rad(TM)$ is of rank r, where $m \geq 2$ and $1 \leq r \leq n$. In particular, (M, g) is called a half lightlike submanifold if $n = 2$ and $r = 1$. It is well known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$, where $TM^{\perp}$ is called normal bundle of M in $\overline{M}$. Thus there exist two non-degenerate complementary distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the screen and screen transversal distribution on M respectively. Thus we have

$$TM = Rad(TM) \oplus_{\text{orth}} S(TM) \tag{2.1}$$

and

$$TM^{\perp} = Rad(TM) \oplus_{\text{orth}} S(TM^{\perp}), \tag{2.2}$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum.

Considering the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\overline{M}$, it is easy to see that $TM^{\perp}$ is a subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ to $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate distribution. Clearly $Rad(TM)$ is a subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\overline{g}(L, L) = \epsilon = \pm 1$. For any null section $\xi \in \Gamma(Rad(TM))$, there exists a unique null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, X) = \overline{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the vector subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we have that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{\text{orth}} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of M with respect to the chosen screen distribution $S(TM)$ respectively. Then $T\overline{M}$ is decomposed as following

$$\begin{aligned} T\overline{M} &= TM \oplus tr(TM) = Rad(TM) \oplus tr(TM) \oplus_{\text{orth}} S(TM) \\ &= Rad(TM) \oplus ltr(TM) \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^{\perp}). \end{aligned} \tag{2.3}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). For any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, $\xi \in \Gamma(Rad(TM))$ and $L \in \Gamma(S(TM)^{\perp})$, the Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + D_1(X,Y)N + D_2(X,Y)L, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \rho_1(X)N + \rho_2(X)L, \tag{2.5}$$

$$\overline{\nabla}_X L = -A_L X + \varepsilon_1(X)N + \varepsilon_2(X)L, \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X Y + E(X,PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -A^*_{\xi} X + u_1(X)\xi, \tag{2.8}$$

respectively, where ∇ and ∇^* are induced connection on TM and $S(TM)$ respectively, D_1 and D_2 are called local second fundamental forms of M, E is called the local second fundamental form of $S(TM)$. A_N, A^*_{ξ} and A_L are linear shape operators on TM of lightlike transversal bundle, radical bundle and screen transversal bundle respectively. ρ_1, ρ_2, ε_1, ε_2 and u_1 are 1-forms on M. Noticing that the following equation holds

$$(\nabla_X g)(Y,Z) = D_1(X,Y)\eta(Z) + D_1(X,Z)\eta(Y), \tag{2.9}$$

for all $X, Y, Z \in \Gamma(TM)$, where $\eta(X) = \overline{g}(X,N)$. So the induced connection ∇ on M is torsion free but is not a metric tensor, it is easy to verify that the induced connection ∇^* on $S(TM)$ is metric. D_1 and D_2 are both symmetric tensors on $\Gamma(TM)$.

From the above statements it is easy to check $\varepsilon_2 = 0$ and the following equations:

$$D_1(X,\xi) = 0, \quad D_1(X,PY) = g(A^*_{\xi}X, PY), \quad \overline{g}(A^*_{\xi}X, N) = 0, \tag{2.10}$$

$$\epsilon D_2(X,Y) = g(A_L X, Y) - \varepsilon_1(X)\eta(Y), \quad \overline{g}(A_L X, N) = \epsilon\rho_2(X), \tag{2.11}$$

$$E(X,PY) = g(A_N X, PY), \quad \overline{g}(A_N X, N) = 0, \quad u_1(X) = -\rho_1(X) \tag{2.12}$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$. From the above equations we also see that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to D_1 and D_2 respectively and A^*_ξ is self-adjoint on M and satisfies

$$A^*_\xi \xi = 0. \tag{2.13}$$

Denote by $\overline{R}$ and R the curvature tensor of semi-Riemannian connection $\overline{\nabla}$ of $\overline{M}$ and the induced connection ∇ of M respectively, we obtain the following Gauss-Codazzi equations for M and $S(TM)$.

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,PW) = & \; g(R(X,Y)Z,PW) + D_1(X,Z)E(Y,PW) - D_1(Y,Z)E(X,PW) \\ & + \epsilon D_2(X,Z)D_2(Y,PW) - \epsilon D_2(Y,Z)D_2(X,PW),\end{aligned} \tag{2.14}$$

$$\overline{g}(\overline{R}(X,Y)Z,N) = \overline{g}(R(X,Y)Z,N) + \epsilon\rho_2(Y)D_2(X,Z) - \epsilon\rho_2(X)D_2(Y,Z), \tag{2.15}$$

$$\overline{g}(\overline{R}(X,Y)Z,\xi) = g(R(X,Y)Z,\xi) + \varepsilon_1(X)D_2(Y,PZ) - \varepsilon_1(Y)D_2(X,PZ), \tag{2.16}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)PZ,u) = & \; \epsilon\big((\nabla_X D_2)(Y,PZ) - (\nabla_X D_2)(Y,PZ)\big) \\ & + \rho_2(X)D_1(Y,PZ) - \rho_2(Y)D_1(X,PZ)\end{aligned} \tag{2.17}$$

for any $X, Y, Z, W \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$.

The Ricci curvature tensor of $\overline{M}$ denoted by $\overline{\mathrm{Ric}}$ is defined by

$$\overline{\mathrm{Ric}}(X,Y) = \mathrm{trace}\{Z \to \overline{R}(X,Z)Y\}, \tag{2.18}$$

where $\overline{R}(X,Y,Z,W) = \overline{g}(\overline{R}(X,Y)Z,W)$. Locally, $\overline{\mathrm{Ric}}(X,Y)$ is given by

$$\overline{\mathrm{Ric}}(X,Y) = \sum_i \epsilon_i \overline{g}(\overline{R}(X,E_i)Y,E_i), \tag{2.19}$$

where $\{E_1, E_2, \cdots, E_{m+2}\}$ is a local semi-orthonormal frame fields of $T\overline{M}$ and $\overline{g}(E_i,E_i) = \epsilon_i$. In particular, if $\overline{\mathrm{Ric}}(X,Y) = k\overline{g}(X,Y)$ for any $X, Y \in \Gamma(TM)$, then $\overline{M}$ is called an Einstein manifold, where k is a smooth function on $\overline{M}$.

3. Co-screen conformal half lightlike submanifolds

In this section, we consider a class of half lightlike submanifolds with co-screen conformal geometric condition defined as following.

Definition 3.1. Let M be a half lightlike submanifold of a semi-Riemannian manifold, M is called co-screen locally (resp. globally) conformal if on any coordinate neighborhood $\mathscr{U}$ $(resp.\ \mathscr{U} = M)$ there exists a non-zero smooth function ϕ such that for any null transversal vector field $N \in \Gamma(ltr(TM))$ the relation

$$A_N X = \phi A_L X, \quad \forall X \in \Gamma(TM) \tag{3.1}$$

holds, where L is a unit vector field of screen transversal bundle of M.

In the sequel, by co-screen conformal we shall mean co-screen globally conformal unless otherwise specified. From (2.12) we know the shape operator A_N is $S(TM)$-valued, thus for a co-screen conformal half lightlike submanifold we have $\rho_2 = 0$ following from (2.11). Also, we have the following theorem to characterize co-screen conformal.

Theorem 3.2. *Let M be a half lightlike submanifold of a semi-Riemannian manifold, then M is co-screen conformal if and only if*

$$E(X,PY) = \epsilon\phi D_2(X,PY) \text{ and } \rho_2(X) = 0,\ \forall X,Y \in \Gamma(TM),$$

where ϕ is a non-zero smooth function on M.

Proof. If M is a co-screen conformal half lightlike submanifold, then it follows from (2.11) and (2.12) that

$$E(X,PY) = g(A_N X,PY) = \phi g(A_L X,PY) = \epsilon\phi D_2(X,PY).$$

Conversely, from (2.11) we know that if $\rho_2(X) = \epsilon\overline{g}(A_L X,N) = 0$, the shape operator A_L is $S(TM)$-valued. Then $E(X,PY) = \epsilon\phi D_2(X,PY)$ implies that $A_N = \phi A_L$, as $S(TM)$ is a non-degenerate distribution. Which completes the proof. □

Denote by $\mathscr{S}$ the first derivative of a screen distribution $S(TM)$ given by [4]

$$\mathscr{S} = \operatorname{Span}\{[X,Y]_x,\ X_x, Y_x \in S(TM_x),\ x \in M\}, \tag{3.2}$$

where $[\ ,\ ]$ denotes the Lie-bracket. If $S(TM)$ is integrable, then $\mathscr{S}$ is sub-bundle of $S(TM)$. Then we have the following lemma.

Lemma 3.3. *Let $(M,g,S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $\overline{M}^{m+2}$ with $m > 1$. Suppose the sub-bundle F of $D^{\perp}$ admits a covariant constant non-null vector field. Then, with respect to a section ξ of $Rad(TM)$, M admits an integrable screen $S(TM)$. Moreover, if the first derivative $\mathscr{S}$ defined by (3.2) coincides with an integral screen distribution $S(TM)$, then, $S(TM)$ is a unique screen of M, up to an orthogonal transformation with a unique lightlike transversal vector bundle and invariant screen second fundamental form.*

Let M be a co-screen conformal half lightlike submanifold, from (2.7) we have

$$\nabla_X PY = \nabla^*_X PY + \epsilon\phi D_2(X,PY)\xi, \quad \forall X,Y \in \Gamma(TM). \tag{3.3}$$

Since the induced connection ∇ on M is torsion-free, it follows from (2.4) and (3.3) that

$$\begin{aligned}\overline{g}([X,Y],N) &= \overline{g}(\nabla_X Y,N) - \overline{g}(\nabla_Y X,N)\\ &= \epsilon\phi D_2(X,Y)\overline{g}(\xi,N) - \epsilon\phi D_2(Y,X)\overline{g}(\xi,N)\end{aligned}$$

$$= \epsilon\phi\big(D_2(X,Y) - D_2(Y,X)\big), \quad \forall X, Y \in \Gamma(S(TM)). \tag{3.4}$$

Noticing that D_2 is symmetric, from (3.4) we know $\overline{g}([X,Y],N) = 0$ for any $X, Y \in \Gamma(S(TM))$. Hence $S(TM)$ is integrable. Now, by Lemma 3.3 we show that a co-screen conformal half lightlike submanifold admits a unique screen distribution.

Theorem 3.4. *Let $(M, g, S(TM))$ be a co-screen conformal half lightlike submanifold of a semi-Riemannian manifold $(\overline{M}^{m+2}, \overline{g})$. Then, any screen distribution $S(TM)$ of M is integrable. Moreover, if the first derivative $\mathscr{S}$ defined by (3.2) coincides with $S(TM)$, then $S(TM)$ is a unique screen of M, up to an orthogonal transformation with a unique lightlike transversal vector bundle and invariant screen second fundamental form.*

Definition 3.5 ([6])**.** Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $\overline{M}$, then M is said to be totally umbilical if there exist smooth functions H_1 and H_2 on M such that

$$D_1(X,Y) = H_1 g(X,Y),\ D_2(X,Y) = H_2 g(X,Y), \quad \forall X, Y \in \Gamma(TM).$$

In particular, if $H_1 = 0$ and $H_2 = 0$, M is said to be totally geodesic.

Definition 3.6 ([6])**.** Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $\overline{M}$, then M is said to be minimal if the trace of the second fundamental form of M restricted on $S(TM)$ vanishes and $\varepsilon_1(\xi) = 0$.

By the above two definitions, we obtain some relationships between the geometry of $S(TM)$ and M as following.

Theorem 3.7. *Let $(M, g, S(TM))$ be a co-screen conformal half lightlike submanifold of a semi-Riemannian manifold $(\overline{M}^{m+2}, \overline{g})$ with a leaf M' of $S(TM)$. Then*

1. *M is totally geodesic,*
2. *M is totally umbilical,*
3. *M is minimal,*

if and only if M' is so immersed as a submanifold of $\overline{M}$ and ϵ_1 vanishes on M.

Proof. Suppose that M is co-screen conformal, from (2.4) and (2.7) we know

$$\overline{\nabla}_X Y = \nabla^*_X Y + \epsilon\phi D_2(X,Y)\xi + D_1(X,Y)N + D_2(X,Y)L, \tag{3.5}$$

for any $X, Y \in \Gamma(TM')$. We see from Theorem 3.4 that a co-screen half lightlike submanifold has an integrable screen distribution, then a leaf of $S(TM)$ is a semi-Riemannian submanifold. Thus, from (3.5) we have

$$\overline{\nabla}_X Y = \nabla'_X Y + h'(X,Y), \tag{3.6}$$

where ∇' and h' denote the second fundamental form and the Levi-Civita connection of M' in $\overline{M}$ respectively. Also, from (3.5) and (3.6) we have

$$h'(X,Y) = (\epsilon\phi\xi + L)D_2(X,Y) + D_1(X,Y)N. \tag{3.7}$$

1. If M is totally geodesic, which means that $D_1 = 0$ and $D_2 = 0$. Then we know that M' is geodesic in $\overline{M}$ from (3.7). Conversely, if $h'(X,Y) = 0$ for any $X, Y \in \Gamma(TM')$ then we have $D_1 = D_2 = 0$, as $\{\xi, N, u\}$ are linearly independent.
2. If M is totally umbilical, by the Definition 3.5 we have that $h'(X,Y) = (\epsilon\phi\xi + L)D_2(X,Y) + D_1(X,Y)N = g(X,Y)H^*$, where $H^* := \alpha\xi + \beta N + \gamma L$ is mean curvature vector field of M' in $\overline{M}$. Therefor, we get $D_1(X,Y) = \beta g(X,Y)$ and $D_2(X,Y) = \gamma g(X,Y)$. Conversely, if M is totally umbilical in $\overline{M}$, then we have $D_1(X,Y) = H_1 g(X,Y)$ and $D_2(X,Y) = H_2 g(X,Y)$. Thus, from (3.7) it is easy to see that M is umbilical in $\overline{M}$.
3. If M is minimal, noticing that $D_1(\xi, X) = 0$ for any $X \in \Gamma(TM)$ we have $\mathrm{tr}|_{S(TM)} D_1 = \mathrm{tr}|_{S(TM)} D_2 = 0$ and $\varepsilon_1(\xi) = 0$, so it is easy to get $\mathrm{tr}|_{S(TM)} h' = 0$ following from (3.7). Conversely, M' is minimal in $\overline{M}$ implies that $\mathrm{tr}|_{S(TM)} h' = 0$, together with the equivalence $\varepsilon_1(X) = 0 \Leftrightarrow D_2(\xi, PX) = D_2(PX, \xi) = D_2(\xi,\xi) = 0$, then we know M is minimal in $\overline{M}$. Which completes the proof. □

Definition 3.8 ([1])**.** A lightlike submanifold M is said to be irrotational if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$, where $\xi \in \Gamma(Rad(TM))$.

For a half lightlike submanifold M, it follows from (2.10) that the above definition is equivalent to $D_2(X,\xi) = -\epsilon\varepsilon_1(X) = 0$ for any $X \in \Gamma(TM)$. Using Theorem 3.7, then we have the following result.

Corollary 3.9. *Let $(M, g, S(TM))$ be an irrotational co-screen conformal half lightlike submanifold of a semi-Riemannian manifold $(\overline{M}^{m+2}, \overline{g})$ with a leaf M' of $S(TM)$. Then*

1. *M is totally geodesic,*
2. *M is totally umbilical,*
3. *M is minimal,*

if and only if M' is so immersed as a submanifold of $\overline{M}$.

Theorem 3.10. *Let $(M, g, S(TM))$ be a co-screen conformal half lightlike submanifold of semi-Riemannian manifold $(\overline{M}^{m+2}, \overline{g})$. Then the following assertions are equivalent.*

(1) *Any leaf of $S(TM)$ is totally geodesic on M.*

(2) *M is a lightlike product manifold of M' and F, where M' is a leaf of $S(TM)$ and F is a null curve of M.*

(3) *D_2 vanishes identically on $S(TM)$.*

Proof. It follows from Theorem 4.4.9 of [7] that

$$g(\nabla_\xi \xi, X) = 0, \quad \forall X \in \Gamma(S(TM)). \tag{3.8}$$

Since $\overline{\nabla}$ is a metric connection on $\overline{M}$, from (2.4) and (2.5) we have that

$$\begin{aligned}\overline{g}(\nabla_X Y, N) &= \overline{g}(\overline{\nabla}_X Y, N) = \overline{\nabla}_X \overline{g}(Y, N) - \overline{g}(Y, \overline{\nabla}_X N)\\ &= -\overline{g}(Y, -A_N X + \rho_1(X)N)\\ &= g(A_N X, Y) = \epsilon\phi D_2(X, Y), \quad \forall X, Y \in \Gamma(S(TM)).\end{aligned} \tag{3.9}$$

Thus, the equivalence between (1) and (2) follows from (2.8) and (3.9). If M is a lightlike product manifold of M' and F, then any leaf of $S(TM)$ is parallel. So $D_2(X, Y) = 0$ for any $X, Y \in \Gamma(S(TM))$ following from (3.9). Conversely, if D_2 vanishes identically on $S(TM)$, by using (3.9) we know that a leaf of $S(TM)$ is parallel, and from (2.8) we obtain (2). Which proves the proof. □

4. Induced Ricci curvature tensor

Let M be a $(m+1)$-dimensional co-screen conformal half lightlike submanifold of semi-Riemannian space form $\overline{M}(c)$ and $S(TM) = \text{span}\{e_1, e_2, \ldots, e_m\}$, where $\{e_i\}$ is a locally orthogonal frame fields of $\Gamma(S(TM))$ and $g(e_i, e_i) = \epsilon_i$. Then from the Gauss-Codazzi equations we have

$$\begin{aligned}\overline{g}(\overline{R}(X, e_i)Y, e_i) = {}& g(R(X, e_i)Y, e_i) + \epsilon\phi D_1(X, Y)D_2(e_i, e_i) - \epsilon\phi D_1(e_i, Y)D_2(e_i, X)\\ &+ \epsilon D_2(X, Y)D_2(e_i, e_i) - \epsilon D_2(e_i, Y)D_2(e_i, X),\end{aligned} \tag{4.1}$$

and

$$\overline{g}(\overline{R}(X, \xi)Y, N) = \overline{g}(R(X, \xi)Y, N). \tag{4.2}$$

By the above equations and the definition of Ricci curvature tensor, we have

$$\begin{aligned}\text{Ric}(X, Y) &= \sum_i \epsilon_i g(R(X, e_i)Y, e_i) + \overline{g}(R(X, \xi)Y, N)\\ &= \sum_i \epsilon_i \overline{g}(\overline{R}(X, e_i)Y, e_i) + \overline{g}(\overline{R}(X, \xi)Y, N) - \epsilon\phi D_1(X, Y)D_2(e_i, e_i)\\ &\quad + \epsilon\phi D_1(e_i, Y)D_2(e_i, X) - \epsilon D_2(X, Y)D_2(e_i, e_i) + \epsilon D_2(e_i, Y)D_2(e_i, X)\\ &= (1-m)c g(X, Y) - \epsilon\phi D_1(X, Y)D_2(e_i, e_i) + \epsilon\phi D_1(e_i, Y)D_2(e_i, X)\\ &\quad - \epsilon D_2(X, Y)D_2(e_i, e_i) + \epsilon D_2(e_i, Y)D_2(e_i, X).\end{aligned} \tag{4.3}$$

Noticing that D_1 and D_2 are symmetric on $\Gamma(TM)$, so we have the following theorem.

Theorem 4.1. *Let M be a $(m+1)$-dimensional co-screen conformal half lightlike submanifold of semi-Riemannian space form $\overline{M}(c)$, then the induced Ricci curvature tensor of M is symmetric.*

In particular, we have the following corollary to characterize the Einstein half lightlike surface.

Corollary 4.2. *Let M be an irrotational co-screen conformal half lightlike surface of 4-dimensional semi-Riemannian space form $\overline{M}(c)$, then the induced Ricci curvature tensor is symmetric. Moreover, the surface is an Einstein surface.*

Proof. From (4.3) we know that

$$\begin{aligned}\mathrm{Ric}(X,Y) &= -cg(X,Y)+\epsilon\phi\big(D_1(e,PY)D_2(e,PX)-D_1(PX,PY)D_2(e,e)\big)\\ &\quad +\epsilon\big(D_2(e,PY)D_2(e,PX)-D_2(PX,PY)D_2(e,e)\big)\\ &= -cg(X,Y),\end{aligned}\tag{4.4}$$

where e is a unit vector field of $S(TM)$. Which proves the corollary. □

Recall the following notion of null sectional curvature [11, 7]. Let $x\in M$ and ξ be a null vector of T_xM. A plane H of T_xM is called a null plane directed by ξ if it contains ξ, $g_x(\xi,W)=0$ for any $W\in H$ and there exists $W_o\in H$ such that $g_x(W_o,W_o)\neq 0$. Thus the null sectional curvature of H with respective to ξ and the induced connection ∇ of M, is defined as a real number

$$K_\xi(H)=\frac{g_x(R(W,\xi)\xi,W)}{g_x(W,W)},$$

where $W\neq 0$ is any vector in H independent with ξ. Note from [12] that an $n(n\geq 3)$-dimensional Lorentzian manifold is of constant curvature if and only if its null sectional curvatures are everywhere zero.

Theorem 4.3. *Let M be a co-screen conformal half lightlike submanifold of semi-Riemannian space form $\overline{M}(c)$, then the null sectional curvature of M is given by*

$$K_\xi(H)=\epsilon\big(D_2(\xi,\xi)D_2(PW,PW)-D_2(\xi,PW)^2\big).$$

Proof. It follows from (2.14) that

$$\begin{aligned}K_\xi(H) &= \overline{g}(\overline{R}(W,\xi)\xi,W)+\epsilon\phi D_1(\xi,\xi)D_2(W,PW)-\epsilon\phi D_1(W,\xi)D_2(\xi,PW)\\ &\quad +\epsilon D_2(\xi,\xi)D_2(W,PW)-\epsilon D_2(W,\xi)D_2(\xi,PW)\\ &= \epsilon\big(D_2(\xi,\xi)D_2(PW,PW)-D_2(\xi,PW)^2\big).\end{aligned}\tag{4.5}$$

Then the proof is completed. □

From Theorem 4.3 it is easy to get the following corollary.

Corollary 4.4. *Let M be co-screen conformal half lightlike submanifold of semi-Riemannian space form $\overline{M}(c)$, then the null sectional curvature of M vanishes if D_2 vanishes on M.*

Theorem 4.5. *Let $(M, g, S(TM))$ be a co-screen conformal half lightlike submanifold of $\overline{M}(c)$ with $D_2 = 0$. Then M is flat if and only if a leaf M' of $S(TM)$ is flat and $c = 0$.*

Proof. Suppose that M is flat, which means the induced semi-Riemannian curvature tensor vanishes on M. It follows from (2.15) that

$$\begin{aligned}\overline{g}(R(X,Y)PZ,N) &= \overline{g}(\overline{R}(X,Y)PZ,N)\\ &= cg(Y,PZ)\overline{g}(X,N) - cg(X,PZ)\overline{g}(Y,N) = 0,\ \forall X,Y,Z \in \Gamma(TM). \quad (4.6)\end{aligned}$$

Thus, for $X = \xi$ in (4.6) we derive $cg(Y, PZ) = 0$ for any $Y, Z \in \Gamma(TM)$ and hence $c = 0$. Also, it follows from (2.4), (2.7) and Theorem 3.2 that

$$\begin{aligned}g(R(X,Y)PZ,PW) &= g(R^*(X,Y)PZ,PW) + \epsilon\phi D_1(Y,PW)D_2(X,PZ)\\ &\quad -\epsilon\phi D_1(X,PW)D_2(Y,PW). \quad (4.7)\end{aligned}$$

Together with $D_2 = 0$ and $R = 0$ in (4.7), then we have $R^* = 0$.

Conversely, if $c = 0$ then from (4.6) we know that $R(X,Y)PZ \in \Gamma(S(TM))$. Noting that M' is flat and $D_1 = 0$, then from (4.7) we obtain $g(R(X,Y)PZ,PW) = 0$. Thus, we get $R(X,Y)PZ = 0$ for any $X,Y,Z \in \Gamma(TM)$. Similarly, it follows from (2.15) that $\overline{g}(R(X,Y)\xi,N) = \overline{g}(\overline{R}(X,Y)\xi,N) = 0$. Which means that $R(X,Y)\xi \in \Gamma(S(TM))$.

On the other hand, from (2.14) and $c = 0$ we have

$$\begin{aligned}g(R(X,Y)\xi,PW) &= g(\overline{R}(X,Y)\xi,PW) + \epsilon\phi D_1(Y,\xi)D_2(X,PW) - \epsilon\phi D_1(X,\xi)D_2(Y,PW)\\ &\quad +\epsilon D_2(Y,\xi)D_2(X,PW) - \epsilon D_2(X,\xi)D_2(Y,PW)\\ &= \epsilon D_2(Y,\xi)D_2(X,PW) - \epsilon D_2(X,\xi)D_2(Y,PW). \quad (4.8)\end{aligned}$$

Using $D_2 = 0$, we get $R(X,Y)\xi = 0$ following from (4.8). Which proves the proof. □

5. Totally umbilical submanifolds

For a co-screen conformal half lightlike submanifold M, it follows from Theorem 3.4 that $S(TM)$ is integrable. Then for a leaf M' of $S(TM)$, it is easy to see that M' is totally umbilical if and only if

$$D_2(X,PY) = \frac{\epsilon}{\phi} g(X,PY), \quad \forall X,Y \in \Gamma(TM). \quad (5.1)$$

Theorem 5.1. *Let M be a co-screen conformal half lightlike submanifold of semi-Riemannian manifold space form $(\overline{M}(c), \overline{g})$. Suppose that $S(TM)$ is integral and a leaf M' of $S(TM)$ is totally umbilical, then*

$$D_1(X,Y) = -\frac{\xi(\phi)}{\phi} g(X,Y), \ \forall X,Y \in \Gamma(TM). \tag{5.2}$$

Proof. Taking a differentiation on both sides of (5.1), we have

$$(\nabla_X D_2)(Y, PZ) = \frac{\epsilon}{\phi}(\nabla_X g)(Y, PZ) + \epsilon X(\frac{1}{\phi}) g(Y, PZ). \tag{5.3}$$

Noticing that $\overline{M}$ is a space form and using Theorem 3.2, it follows from (2.17) that

$$(\nabla_X D_2)(Y, PZ) - (\nabla_X D_2)(Y, PZ) = 0. \tag{5.4}$$

Substituting (5.4) into (5.3) and using (2.9), we have

$$D_1(X,Z)\eta(Y) - D_1(Y,Z)\eta(X) = \phi Y(\frac{1}{\phi}) g(X,Z) - \phi X(\frac{1}{\phi}) g(Y,Z). \tag{5.5}$$

Replacing Y by ξ in the above equation and using (2.10), we have

$$D_1(X,Z) = \phi\xi(\frac{1}{\phi}) g(X,Z), \quad \forall X,Z \in \Gamma(TM). \tag{5.6}$$

Which proves the theorem. □

Theorem 5.2. *Let M be an irrotational half lightlike submanifold of semi-Riemannian manifold $\overline{M}$ with $\rho_2 = 0$. Suppose that $S(TM)$ is integral and any leaf M' of $S(TM)$ is totally umbilical immersed in $\overline{M}$ as a co-dimensional 3 non-degenerate submanifold with $\alpha\gamma \neq 0$. Then M is co-screen conformal if and only if $E(\xi, PX) = 0$ for $\xi \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$, where α and γ are components of the mean curvature vector of the leaf, in the direction to ξ and u respectively.*

Proof. Denote by M' a leaf of $S(TM)$, then from (2.4) and (2.7) we have that

$$\nabla_X Y = \nabla^*_X Y + E(X,Y)\xi + D_1(X,Y)N + D_2(X,Y)L, \ \forall X,Y \in \Gamma(TM'). \tag{5.7}$$

Suppose that the mean curvature vector field H^* of M' in $\overline{M}$ is $H^* = \alpha\xi + \beta N + \gamma L$. Since M' is totally umbilical in $\overline{M}$, we have

$$E(X,Y)\xi + D_1(X,Y)N + D_2(X,Y)L = g(X,Y)(\alpha\xi + \beta N + \gamma L). \tag{5.8}$$

As ξ, N, L are linearly independent, then (5.8) is equivalent to the following equations.

$$E(X,Y) = \alpha g(X,Y), \quad D_1(X,Y) = \beta g(X,Y), \quad D_2(X,Y) = \gamma g(X,Y). \tag{5.9}$$

Therefore we get $E(X,Y) = \frac{\alpha}{\gamma}D_2(X,Y)$ for any $X,Y \in \Gamma(TM')$. On the other hand, with the definition of irrotational submanifold we see $E(\xi,PX) = D_2(\xi,PX) = 0$ for any $X \in \Gamma(TM)$. It follows from Theorem 3.2 that M is co-screen conformal. Conversely, if M is co-screen conformal, then from Theorem 3.2 we have $E(X,PY) = \epsilon\phi D_2(X,PY)$. So $E(\xi,PX) = 0$ follows from the Definition 3.8. Which proves the proof. □

Theorem 5.3. *Let M be a co-screen conformal half lightlike submanifold of semi-Riemannian manifold $\overline{M}$. Then M is totally umbilical if and only if*

$$P(A^*_\xi X) = H_1 PX \textit{ and } \varepsilon_1(X) = 0,\ \forall X \in \Gamma(TM), \tag{5.10}$$

and a leaf M' of $S(TM)$ is totally umbilical in M.

Proof. Suppose that M is umbilical, then we have $D_1(X,Y) = H_1 g(X,Y)$ and $D_2(X,Y) = H_2 g(X,Y)$. Noticing Theorem 3.2, we have

$$E(X,PY) = \epsilon\phi D_2(X,PY) = \epsilon\phi H_2 g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{5.11}$$

Which means that M' is totally umbilical in M. On the other hand, it follows from (2.9) that

$$D_1(X,Y) = g(A^*_\xi X, Y) = H_1 g(X,Y), \quad \forall X,Y \in \Gamma(TM). \tag{5.12}$$

Then we have $A^*_\xi = H_1 PX$ as A^*_ξ is $S(TM)$-valued. Conversely, if M' is umbilical in M. Then we have $E(X,Y) = H_3 g(X,Y)$ for any $X,Y \in \Gamma(TM')$, where H_3 is a smooth function on M'. By (5.11) we have $D_2(X,Y) = \frac{\epsilon H_3}{\phi} g(X,Y)$ for any $X,Y \in \Gamma(TM')$. Using (2.11) we see $\varepsilon_1(X) = D_2(\xi,X) = 0$. Thus, we have $D_2(X,Y) = \frac{\epsilon H_3}{\phi} g(X,Y)$ for any $X,Y \in \Gamma(TM)$. It follows from (5.12) that M is umbilical in $\overline{M}$. Which proves the theorem. □

Corollary 5.4. *Let M be an irrotational co-screen conformal half lightlike submanifold of semi-Riemannian manifold $\overline{M}$. Then M is totally umbilical if and only if*

$$P(A^*_\xi X) = H_1 PX, \quad \forall X \in \Gamma(TM) \tag{5.13}$$

and a leaf M' of $S(TM)$ is totally umbilical in M.

Theorem 5.5. *Let M be an irrotational co-screen conformal totally umbilical half lightlike submanifold of semi-Riemannian manifold $\overline{M}$. Suppose that M' is a leaf of $S(TM)$, then*

1. *M' is totally umbilical in $\overline{M}$.*
2. *M is totally geodesic if and only if M' is totally geodesic immersed in M.*

Proof. Since M is co-screen conformal, then $S(TM)$ is integrable. For a leaf M' of $S(TM)$, from (2.4) and (2.7) we have that

$$\overline{\nabla}_X Y = \nabla^*_X Y + \epsilon\phi D_2(X,Y)\xi + D_1(X,Y)N + D_2(X,Y)L, \tag{5.14}$$

where $X, Y \in \Gamma(TM')$. By Theorem 3.2 we have $E(X,Y) = \epsilon\phi D_2(X,Y) = \epsilon\phi H_2 g(X,Y)$ for any $X, Y \in \Gamma(TM')$, which means that M' is totally umbilical in $\overline{M}$.

As ξ, N and L are linearly independent, then $\epsilon\phi D_2(X,Y)\xi + D_1(X,Y)N + D_2(X,Y)L = 0$ for any $X, Y \in \Gamma(TM')$ is equivalent to $D_1(X,Y) = D_2(X,Y) = 0$ for any $X, Y \in \Gamma(TM')$. Also, from (2.10) we know that $D_1(\xi, X) = 0$ for any $X \in \Gamma(TM)$. Noticing Definition 3.8, we have $D_1(X,Y) = D_2(X,Y) = 0$ for any $X, Y \in \Gamma(TM)$. Which completes the proof. □

Theorem 5.6. *Let $(M, g, S(TM))$ be a co-screen conformal totally umbilical half lightlike submanifold of a semi-Riemannian space form $\overline{M}(c)$ and M' be a leaf of $S(TM)$. If $dim(M') > 2$, then M' is a semi-Riemannian space form if and only if ϕ is a constant.*

Proof. It follows from (2.14) and Theorem 3.2 that

$$\begin{aligned} g(R(X,Y)Z, PW) &= g(\overline{R}(X,Y)Z, PW) + \epsilon\phi D_1(Y,Z)D_2(X,PW) - \epsilon\phi D_1(X,Z)D_2(Y,PW) \\ &\quad + \epsilon D_2(Y,Z)D_2(X,PW) - \epsilon D_2(X,Z)D_2(Y,PW) \\ &= (c + \epsilon\phi H_1 H_2 + \epsilon H_2^2)\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big), \end{aligned} \tag{5.15}$$

where $X, Y, Z, W \in \Gamma(TM)$. On the other hand, it follows from (2.7) and Theorem 3.2 that

$$\begin{aligned} g(R(X,Y)PZ, PW) &= g(R^*(X,Y)PZ, PW) + E(X,PZ)D_1(Y,PW) - E(Y,PZ)D_1(X,PW) \\ &= g(R^*(X,Y)PZ, PW) + \epsilon\phi H_1 H_2\big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big), \end{aligned} \tag{5.16}$$

where $X, Y, Z, W \in \Gamma(TM)$. Thus from (5.16) and (5.17) we obtain

$$g(R^*(X,Y)Z, W) = (c + 2\epsilon\phi H_1 H_2 + \epsilon H_2^2)\big(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\big), \tag{5.17}$$

where $Z, W \in \Gamma(S(TM))$. Which completes the proof. □

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