ON A CHARACTERIZATION OF GENERALIZED ERLANG-TRUNCATED EXPONENTIAL DISTRIBUTION THROUGH DISTRIBUTIONAL PROPERTIES OF DUAL GENERALIZED ORDER STATISTICS

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Abstract. Generalized Erlang-truncated exponential distribtion $F(x) = [1 - e^{-\beta(1-e^{-\lambda})x}]$ has been characterized through translation of two non-adjacent dual generalized order statistics (dgos) and then the characterizing results are obtained for generalized Pareto distribution through dilation of dual generalized order statistics (dgos) and generalized power function distribution through contraction of non-adjacent generalized order statistics (gos). Further, the results are deduced for order statistics, lower record statistics, upper record statistics and adjacent generalized order statistics and dual generalized order statistics.

1. Introduction

Kamps [6] introduced the concept of generalized order statistics (gos) as follows: Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variables (rv) with the absolutely continuous distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (a, b)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + (n-r) + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. If $m_1 = m_2 = \cdots = m_{n-1} = m$, then $X(r, n, m, k)$ is called the rth $m - gos$ and its pdf is given as:

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!} [\bar{F}(x)]^{\gamma_i^{(n)}-1} g_m^{r-1}(F(x)) f(x), \quad a < x < b \tag{1.1}$$

where

$$C_{r-1}^{(n)} = \Pi_{i-1}^r \gamma_i^{(n)}, \quad \gamma_i^{(n)} = k + (n-i)(m+1), \quad 1 \leq r \leq n \tag{1.2}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0, 1).$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \; m \neq -1 \\ -\log(1-x) & , \; m = -1 \end{cases} \tag{1.3}$$

Now since $\lim_{m \to -1} h_m(x) = -\lim_{m \to -1} \frac{(1-x)^{m+1}}{m+1} = -\log(1-x)$, therefore, we will consider $h_m(x)$ $= -\frac{1}{m+1}(1-x)^{m+1}$, for all m, unless otherwise needed.

Based on the generalized order statistics (gos), Burkschat et al. [4] introduced the concept of the dual generalized order statistics ($dgos$) where the pdf of the rth $m-dgos$ $X^*(r, n, m, k)$ is given as

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}^{(n)}}{(r-1)!}[F(x)]^{\gamma_i^{(n)}-1}g_m^{r-1}(F(x))f(x), \quad a < x < b \tag{1.4}$$

which is obtained just by replacing $\bar{F}(x) = 1 - F(x)$ by $F(x)$

where

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}x^{m+1} & , \; m \neq -1 \\ -\log x & , \; m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \; x \in [0, 1).$$

since $-\lim_{m \to -1} \frac{1}{m+1}x^{m+1} \to -\log x$, therefore, we will consider $h_m(x) = -\frac{1}{m+1}x^{m+1}$, for all m, unless otherwise needed.

If support of the distribution $F(x)$ be over (a, b), then by convention, we will write

$$X(0, n, m, k) = a \quad \text{and} \quad X^*(0, n, m, k) = b \tag{1.5}$$

Ahsanullah [1] has characterized uniform distribution under random contraction for adjacent $dgos$. Khan and Shah [7] have characterized distributions using distributional properties of non- adjacent lower records, upper records and order statistics. In this paper, distributional properties of the $dgos$ have been used to characterize a general form of Erlang-truncated exponential distribution for non- adjacent $dgos$ under random translation, dilation and contraction, thus generalizing the results of Ahsanullah [1]. Further, results in terms of lower records, upper records and order statistics are deduced. One may also refer to Alzaid and Ahsanullah [2], Wesolowski and Ahsanullah [8], Beutner and Kamps [3] and Castaño-Martínez et al. [5] for the related results.

It may be seen that if Y is a measurable function of X with the relation

$$Y = h(X)$$

then

(i) $Y^*(r,n,m,k) = h[X^*(r,n,m,k)]$ $\qquad$ (1.6)

(ii) $Y_{r:n} = h(X_{r:n})$ $\qquad$ (1.7)

(iii) $Y_{L(r)} = h(X_{L(r)})$ $\qquad$ (1.8)

if h is an increasing function and

(i) $Y(r,n,m,k) = h[X^*(r,n,m,k)]$ $\qquad$ (1.9)

(ii) $Y_{n-r+1:n} = h(X_{r:n})$ $\qquad$ (1.10)

(iii) $X_{U(r)} = h(X_{L(r)})$ $\qquad$ (1.11)

if h is a decreasing function

where $X_{r:n}$ is the rth order statistics from a sample of size n, $X_{U(r)}$ is the rth upper record and $X_{L(r)}$ is the rth lower record, $X(r,n,m,k)$ is the rth $m-gos$ and $X^*(r,n,m,k)$ is the rth $m-dgos$.

We will denote

(i) $X \sim$ Erlang-truncated $\exp(\beta(\alpha_\lambda))$

if X has an Erlang-truncated exponential distribution with the df

$$F(x) = [1 - e^{-\beta(\alpha_\lambda)x}], \quad 0 \le x < \infty, \ \beta > 0, \lambda > 0 \qquad (1.12)$$

where $\alpha_\lambda = (1 - e^{-\lambda})$

(ii) $X \sim \text{Par}(\beta(\alpha_\lambda))$

if X has a Pareto distribution with the df

$$F(x) = [1 - x^{-\beta(\alpha_\lambda)}], \quad 1 < x < \infty, \ \beta > 0, \lambda > 0 \qquad (1.13)$$

(iii) $X \sim \text{pow}(\beta(\alpha_\lambda))$

if X has a power function distribution with the df

$$F(x) = x^{\beta(\alpha_\lambda)}, \quad 0 < x < 1, \ \beta > 0, \lambda > 0 \qquad (1.14)$$

(iv) $X \sim \text{genErlang-truncated } \exp(\beta(\alpha_\lambda))$

if X has a generalized Erlang-truncated exponential distribution with the df

$$F(x) = [1 - (m+1)e^{-\beta(\alpha_\lambda)x}]^{\frac{1}{m+1}}, \quad \frac{1}{\beta(\alpha_\lambda)} \ln(m+1) < x < \infty, \ \beta > 0 \qquad (1.15)$$

(v) $X \sim \text{genPar}(\beta(\alpha_\lambda))$

if X has a generalized Pareto distribution with the df

$$F(x) = [1-(m+1)x^{-\beta(\alpha_\lambda)}]^{\frac{1}{m+1}}, \quad (m+1)^{\frac{1}{\beta(\alpha_\lambda)}} < x < \infty, \ \beta > 0 \tag{1.16}$$

(iii) $X \sim \text{genpow}(\beta(\alpha_\lambda))$

if X has a generalized power function distribution with the df

$$F(x) = 1-[1-(m+1)x^{\beta(\alpha_\lambda)}]^{\frac{1}{m+1}}, \quad 0 < x < (m+1)^{\frac{1}{\beta(\alpha_\lambda)}}, \ \beta > 0 \tag{1.17}$$

It may further be noted that

$$\text{if} \quad \log X \sim \text{Erlang-truncated} \ \exp(\beta(\alpha_\lambda)) \text{ then } X \sim \text{Par}(\beta(\alpha_\lambda)) \tag{1.18}$$

$$\text{if} \ -\log X \sim \text{Erlang-truncated} \ \exp(\beta(\alpha_\lambda)) \text{ then } X \sim \ \text{pow} \ (\beta(\alpha_\lambda)) \tag{1.19}$$

$$\text{if} \quad \log X \sim \text{genErlang-truncated} \ \exp(\beta(\alpha_\lambda)) \text{ then } X \sim \ \text{genPar} \ (\beta(\alpha_\lambda)) \tag{1.20}$$

$$\text{and if} \ -\log X \sim \text{genErlang-truncated} \ \exp(\beta(\alpha_\lambda)) \text{ then } X \sim \ \text{genpow} \ (\beta(\alpha_\lambda)) \tag{1.21}$$

It has been assumed here throughout that the df is differentiable *w.r.t.* its argument.

2. Characterizing results

Theorem 2.1. *Let $X^*(s,n,m,k)$ be the sth $m-dgos$ from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then*

$$X^*(s-j,n,m,k) \stackrel{d}{=} X^*(s,n,m,k) + Y_{j:s-1}, j = s-r-1, s-r; 1 \leq r < s \leq n \tag{2.1}$$

where $Y_{j:s-1}$ is the jth order statistic from a sample of size $(s-1)$ drawn from Erlang-truncated $\exp(\beta(\alpha_\lambda)$ distribution and is independent of $X^*(s,n,m,k)$ if and only if $X_1 \sim \text{genErlang-truncated}$ $\exp(\beta(\alpha_\lambda))$ and $X \stackrel{d}{=} Y$ denotes that X and Y have the same df

Proof. To prove the necessary part, let the moment generating function (mgf) of $X^*(r,n,m,k)$ be $M_{X^*_{(r)}}(t)$, then

$$X^*(r,n,m,k) \stackrel{d}{=} X^*(s,n,m,k) + Y$$

which implies that

$$M_{X^*_{(r)}}(t) \stackrel{d}{=} M_{X^*_{(s)}}(t) \cdot M_Y(t)$$

Since for the genErlang-truncated $\exp(\beta(\alpha_\lambda))$ distribution.

$$M_{X^*_{(r)}}(t) = \frac{C_{r-1}^{(n)}}{(r-1)!} \frac{1}{(m+1)^{r-\frac{t}{\beta(\alpha_\lambda)}}} \frac{\Gamma\left(r-\frac{t}{\beta(\alpha_\lambda)}\right)\Gamma\left(\frac{\gamma_r^{(n)}}{(m+1)}\right)}{\Gamma\left(r-\frac{t}{\beta(\alpha_\lambda)}+\frac{\gamma_r^{(n)}}{m+1}\right)}$$

Therefore

$$M_Y(t) = \frac{M_{X^*_{(r)}}(t)}{M_{X^*_{(s)}}(t)} = \frac{\Gamma(s)\Gamma(r - \frac{t}{\beta(\alpha_\lambda)})}{\Gamma(r)\Gamma(s - \frac{t}{\beta(\alpha_\lambda)})}$$

But this is the mgf of $Y_{s-r:s-1}$, the $(s-r)^{\text{th}}$ order statistic from a sample of size $(s-1)$ drawn from Erlang-truncated $\exp(\beta(\alpha_\lambda)$ and hence the result.

For the proof of sufficiency part, we have by the convolution method

$$f_{X^*(r,n,m,k)}(x) = \int_0^x f_{X^*(s,n,m,k)}(y) f_{Y_{s-r:s-1}}(x-y)\,dy$$

$$= \frac{\beta(\alpha_\lambda)(s-1)!}{(r-1)!(s-r-1)!} \int_0^x [e^{-\beta(\alpha_\lambda)(x-y)}]^r [1 - e^{-\beta(\alpha_\lambda)(x-y)}]^{s-r-1} f_{X^*(s,n,m,k)}(y)\,dy \quad (2.2)$$

Differentiate both the sides of (2.2) $w.r.t.$ x, to get

$$\frac{d}{dx} f_{X^*(r,n,m,k)}(x)$$

$$= \frac{\beta(\alpha_\lambda)(s-1)!}{(r-1)!(s-r-2)!} \int_0^x \beta(\alpha_\lambda) [e^{-\beta(\alpha_\lambda)(x-y)}]^{r+1} [1 - e^{-\beta(\alpha_\lambda)(x-y)}]^{s-r-2} f_{X^*(s,n,m,k)}(y)\,dy$$

$$- \frac{\beta(\alpha_\lambda)(s-1)!\,r}{(r-1)!(s-r-1)!} \int_0^x \beta(\alpha_\lambda) [e^{-\beta(\alpha_\lambda)(x-y)}]^r [1 - e^{-\beta(\alpha_\lambda)(x-y)}]^{s-r-1} f_{X^*(s,n,m,k)}(y)\,dy$$

$$= \beta(\alpha_\lambda)\,r\,[f_{X^*(r+1,n,m,k)}(x) - f_{X^*(r,n,m,k)}(x)]$$

or, $$f_{X^*(r,n,m,k)}(x) = \beta(\alpha_\lambda)\,r\,[F_{X^*(r+1,n,m,k)}(x) - F_{X^*(r,n,m,k)}(x)] \quad (2.3)$$

Now, since (Ahsanullah [1])

$$F_{X^*(r+1,n,m,k)}(x) - F_{X^*(r,n,m,k)}(x) = \frac{F(x)}{\gamma_{r+1} f(x)} f_{X^*(r+1,n,m,k)}(x) \quad (2.4)$$

Therefore, in view of (1.4), (2.3) and (2.4), we have

$$\frac{(m+1)[F(x)]^m\,f(x)}{[1 - (F(x))^{m+1}]} = \beta(\alpha_\lambda)$$

implying that

$$F(x) = [1 - (m+1)e^{-\beta(\alpha_\lambda)x}]^{\frac{1}{m+1}}$$

and hence the proof.

Remark 2.1. Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{r+j:n} \overset{d}{=} X_{r:n} + Y_{j:n-r}, j = s-r-1, s-r; 1 \le r < s \le n \quad (2.5)$$

where $Y_{j:n-r} \overset{d}{=} X_{j:n-r}$ is independent of $X_{r:n}$ if and only if $X_1 \sim \exp(\beta)$.

This can be established by noting that order statistic appear in the generalized order statistics (gos) model as well as in dual generalized order statistics ($dgos$) model, therefore at $m = 0$ and $\alpha_\lambda = \alpha$, (2.1) may be written as

$$X_{n-s+j+1:n} \stackrel{d}{=} X_{n-s+1:n} + Y_{j:s-1}, j = s - r - 1, s - r; 1 \leq r < s \leq n$$

which implies

$$X_{r+j:n} \stackrel{d}{=} X_{r:n} + Y_{j:n-r}, j = s - r - 1, s - r; 1 \leq r < s \leq n$$

obtained by replacing $(n - s + 1)$ by r and $(n - r + 1)$ by s, as given by Khan and Shah [7].

Remark 2.2. For $\alpha_\lambda = \alpha$, Alzaid and Ahsanullah [2] have proved that

$$X_{r:n} \stackrel{d}{=} X_{r-1:n} + V$$

where $V \sim \exp(n - r + 1)$ *if* and only *if* $X_1 \sim \exp(1)$.

Remark 2.3. For $\alpha_\lambda = \alpha$, Castaño-Martínez et al. [5] have shown that

$$X_{s:n} \stackrel{d}{=} X_{r:n} + V$$

where $V \stackrel{d}{=} -\log W$ with $W \sim \text{Be}(n - r + 1, s - r)$ *if* and only *if* $X_1 \sim \exp(1)$.

Remark 2.4. As $m \to -1$ and $\alpha_\lambda = \alpha$, genexp(β) tends to the Gum(β) and $X^*(r, n, m, k)$ to $X_{L(r)}$, the rth lower records. Therefore we have

$$X_{L(s-j)} \stackrel{d}{=} X_{L(s)} + Y_{j:s-1}, j = s - r - 1, s - r; 1 \leq r < s \leq n \qquad (2.6)$$

where $Y_{j:s-1}$ is the jth order statistic from a sample of size $(s - 1)$ drawn from exp(β) distribution and is independent of $X_{L(s)}$ *if* and only *if* $X_1 \sim \text{Gum}(\beta)$, as obtained by Khan and Shah [7].

Remark 2.5. For $\alpha_\lambda = \alpha$, Alzaid and Ahsanullah [2] have shown that

$$X_{L(r)} \stackrel{d}{=} X_{L(r+1)} + V$$

where $V \sim \exp(r)$ *if* and only *if* $X_1 \sim Gum(1)$.

Corollary 2.2. *Let $X^*(s, n, m, k)$ be the sth $m - dgos$ from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then*

$$X^*(s - j, n, m, k) \stackrel{d}{=} X^*(s, n, m, k) \cdot Y_{j:s-1}, j = s - r - 1, s - r; 1 \leq r < s \leq n \qquad (2.7)$$

where $Y_{j:s-1}$ is the jth order statistic from a sample of size $(s-1)$ drawn from Par $(\beta(\alpha_\lambda)$ distribution and is independent of $X^*(s, n, m, k)$ if and only if $X_1 \sim genPar(\beta(\alpha_\lambda))$.

Proof. Here the product $X^*(s, n, m, k) \cdot Y_{j:s-1}$ in (2.7) is called random dilation of $X^*(s, n, m, k)$ (Beutner and Kamps [3]). Note that if

$$\log X^*(r, n, m, k) \stackrel{d}{=} \log X^*(s, n, m, k) + \log Y_{s-r:s-1}$$

then

$$X^*(r, n, m, k) \stackrel{d}{=} X^*(s, n, m, k) \cdot Y_{s-r:s-1}$$

in view of (1.6), (1.7), (1.18) and (1.20) and the result follows.

Remark 2.6. In case of ordinary order statistics, i.e., at $m = 0$ and $\alpha_\lambda = \alpha$, we have

$$X_{r+j:n} \stackrel{d}{=} X_{r:n} + Y_{j:n-r}, j = s - r - 1, s - r; 1 \le r < s \le n$$

where $Y_{j:n-r} \stackrel{d}{=} X_{j:n-r}$ is independent of $X_{r:n}$ if and only if $X_1 \sim \mathrm{Par}(\beta)$, as obtained by Castaño-Martínez et al. [5] and Khan and Shah [7].

Remark 2.7. As $m \to -1$ and $\alpha_\lambda = \alpha$, we get

$$X_{L(s-j)} \stackrel{d}{=} X_{L(s)} \cdot Y_{j:s-1}, j = s - r - 1, s - r; 1 \le r < s \le n$$

where $Y_{j:s-1}$ is the jth order statistic from a sample of size $(s-1)$ drawn from $\mathrm{Par}(\beta)$ distribution and is independent of $X_{L(s)}$, be the sth lower records if and only if $X_1 \sim \mathrm{inW}(\beta)$.

Corollary 2.3. *Let $X(s, n, m, k)$ be the sth $m - gos$ from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then*

$$X(s - j, n, m, k) \stackrel{d}{=} X(s, n, m, k) \cdot Y_{s-j:s-1}, j = s - r - 1, s - r; 1 \le r < s \le n \qquad (2.8)$$

where $Y_{s-j:s-1}$ is the $(s-j)^{\mathrm{th}}$ order statistic from a sample of size $(s-1)$ drawn from pow $(\beta(\alpha_\lambda)$ distribution and is independent of $X(s, n, m, k)$ if and only if $X_1 \sim genpow(\beta(\alpha_\lambda))$.

Proof. Here the product $X(s, n, m, k) \cdot Y_{s-j:s-1}$ in (2.8) is called random contraction of $X(s - j, n, m, k)$ (Beutner and Kamps [3]). Since

$$-\log X^*(r, n, m, k) \stackrel{d}{=} -\log X^*(s, n, m, k) - \log Y_{s-r:s-1}$$

implies

$$X(r, n, m, k) \stackrel{d}{=} X(s, n, m, k) \cdot Y_{r:s-1}$$

in view of (1.9), (1.10), (1.19) and (1.21) and the result follows. $\qquad \square$

Remark 2.8. For $\alpha_\lambda = \alpha$, Beutner and Kamps [3] have shown that for adjacent generalized order statistics

$$X(r, n, m, k) \stackrel{d}{=} X(r + 1, n, m, k) \cdot V$$

where $V \sim \mathrm{pow}(\beta)$ if and only if $X_1 \sim \mathrm{genpow}(\beta)$.

Remark 2.9. We can get the corresponding characterizing results for the order statistics at $m = 0$ and $\alpha_\lambda = \alpha$ as:

Let $X_{r:n}$ be the rth order statistic from a sample of size n drawn from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{s-j:n} \stackrel{d}{=} X_{s:n} \cdot Y_{s-j:s-1}, j = s-r-1, s-r; 1 \leq r < s \leq n \qquad (2.9)$$

where $Y_{s-j:s-1} \stackrel{d}{=} X_{s-j:s-1}$ is independent of $X_{s:n}$ if and only if $X_1 \sim \text{pow}(\beta)$, as given by Khan and Shah [7].

For adjacent order statistics one may also refer to Ahsanullah [1], Wesolowski and Ahsanullah [8].

Remark 2.10. The corresponding result for the lower records as $m \to -1$ and $\alpha_\lambda = \alpha$ is:

Let $X_{U(s)}$ be the sth upper record from a continuous population with the pdf $f(x)$ and the df $F(x)$, then

$$X_{U(s-j)} \stackrel{d}{=} X_{U(s)} \cdot Y_{s-j:s-1}, j = s-r-1, s-r; 1 \leq r < s \qquad (2.10)$$

where $Y_{s-j:s-1}$ is the $(s-j)^{\text{th}}$ order statistic from a sample of size $(s-1)$ drawn from $\text{pow}(\beta)$ distribution and is of $X_{U(s)}$ if and only if $X_1 \sim Wei\ (\beta)$, as obtained by Khan and Shah [7].

Acknowledgements

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