

ON CERTAIN APPLICATIONS OF DIFFERENTIAL SUBORDINATIONS FOR Φ -LIKE FUNCTIONS

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Abstract. Let $f(z)$ be a normalized analytic function in $\Delta = \{z|z \in \mathbb{C} \text{ and } |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let Φ be an analytic function in a domain containing $f(\Delta)$, with $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\Delta) - \{0\}$. Let $q(z)$ be a fixed analytic function in Δ , $q(0) = 1$. The function f is called Φ -like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta).$$

In this paper, we obtain some sufficient conditions for functions to be Φ -like with respect to $q(z)$.

1. Introduction and Definitions

Let \mathcal{A} be the class of all normalized analytic functions $f(z)$ in the open unit disk $\Delta = \{z|z \in \mathbb{C} \text{ and } |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let Φ be an analytic function in a domain containing $f(\Delta)$, $\Phi(0) = 0$ and $\Phi'(0) > 0$. The function $f \in \mathcal{A}$ is called Φ -like if

$$\Re \frac{zf'(z)}{\Phi(f(z))} > 0 \quad (z \in \Delta).$$

This concept was introduced by Brickman [1] and it was shown that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is Φ -like for some Φ . When $\Phi(w) = w$ and $\Phi(w) = \lambda w$, the function f is starlike and spirallike of type $\arg \lambda$ respectively. Later, Ruscheweyh [13] introduced and studied the following general class of Φ -like functions:

Definition 1.1. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(\omega) \neq 0$ for $\omega \in f(\Delta) - \{0\}$. Let $q(z)$ be a fixed analytic function in Δ , $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ -like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta).$$

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When $\Phi(w) = w$, we denote the class of all Φ -like functions with respect to q by $S^*(q)$.

Li and Owa [4], Lewandowski *et al.* [3] and Ramesha *et al.* [10], Nunokawa *et al.* [6], Ravichandran *et al.* [11], and Padamanbhan [9] have found out sufficient conditions for functions to be starlike or starlike of order α in terms of

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}.$$

Ravichandran [12, Theorem 3, p.44] have obtained a sufficient condition for functions to be in the class $S^*(q)$ which generalize or improve the earlier results. Our first aim in this paper is to obtain a sufficient conditions for a function $f(z)$ to be a Φ -like functions with respect to $q(z)$.

Kim [2] have found out estimates on $C(\alpha)$ for which

$$p(z) \left(1 + \frac{zp'(z)}{p(z)}\right)^\alpha \prec C(\alpha)h(z) \quad (0 \leq \alpha \leq 1)$$

implies $p(z) \prec h(z)$. In fact, they have shown that $\left(\frac{3}{2}\right)^\alpha \leq C(\alpha) \leq 2^\alpha$ for $\alpha \in [0, 1]$ for certain $p(z)$ and $h(z)$. Parvatham [8] considered a similar problem. In this paper, our second aim is to determine the function $h(z)$ such that

$$p(z) \left(1 + \frac{zp'(z)}{p(z)}\right)^\alpha \prec h(z)$$

implies $p(z) \prec q(z)$ and apply this result to derive a sufficient condition for a function to Φ -like with respect to $q(z)$.

2. Subordination Results for Φ -like Functions

In our present investigation, we need the following result of Miller and Mocanu [5]:

Lemma 2.1.(cf. Miller and Mocanu [5, p.132, Theorem 3.4h]) *Let $q(z)$ be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set*

$$Q(z) := zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) := \theta(q(z)) + Q(z).$$

Suppose that

1. $Q(z)$ is starlike univalent in Δ and
2. $\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in \Delta$.

If $p(z)$ is analytic with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \quad (2.2.1)$$

then

$$p(z) \prec q(z)$$

and $q(z)$ is the best dominant.

By appealing to Lemma 2.1, we first prove the following:

Theorem 2.2. *Let $\alpha \neq 0$ be a complex number and $q(z)$ be convex univalent in Δ . Define $h(z)$ by*

$$h(z) := \alpha q^2(z) + (1 - \alpha)q(z) + \alpha zq'(z). \tag{2.2.2}$$

Further assume that

$$\Re \left\{ \frac{1 - \alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0 \quad (z \in \Delta). \tag{2.2.3}$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\Phi(f(z))} \left[1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z [f'(z) - \{\Phi(f(z))\}']}{\Phi(f(z))} \right] \prec h(z), \tag{2.2.4}$$

then

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z)$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \Delta). \tag{2.2.5}$$

Then the function $p(z)$ is analytic in Δ and $p(0) = 1$. Therefore, by making use of (2.2.5), we obtain

$$\begin{aligned} & \frac{zf'(z)}{\Phi(f(z))} \left[1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z [f'(z) - (\Phi(f(z)))']}{\Phi(f(z))} \right] \\ &= p(z) \left[1 + \alpha \left(\frac{zp'(z)}{p(z)} - 1 \right) + \alpha p(z) \right] \\ &= \alpha p^2(z) + (1 - \alpha)p(z) + \alpha zp'(z). \end{aligned} \tag{2.2.6}$$

By using equation (2.2.6) in subordination (2.2.4), we have

$$\alpha p^2(z) + (1 - \alpha)p(z) + \alpha zp'(z) \prec \alpha q^2(z) + (1 - \alpha)q(z) + \alpha zq'(z). \tag{2.2.7}$$

The subordination (2.2.7) is same as (2.2.1) with

$$\phi(\omega) := \alpha \omega^2 + (1 - \alpha)\omega \quad \text{and} \quad \theta(\omega) := \alpha.$$

The result now follows by an application of Lemma 2.1.

Remark 2.1. By taking $\Phi(w) = w$, we obtain [12, Theorem 3, p.44] as a special case of Theorem 2.2.

Remark 2.2. The condition

$$\Re \left\{ \frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0 \quad (z \in \Delta)$$

is satisfied by any convex function which maps Δ onto a convex region in the right-half plane when $0 < \alpha \leq 1$.

Remark 2.3. By taking $\Phi(w) := w$ and $q(z) := (1+Az)/(1+Bz)$, $-1 \leq B < A \leq 1$, in the above Theorem 2.1, we get a sufficient condition for function to be Janowski starlike.

3. Another Sufficient Condition

We need the following Lemma to our main result of this section.

Lemma 3.1. Let $\alpha \neq 0$ be any complex number and $\beta := \max\{0, -\Re\frac{1}{\alpha}\}$. Let $q(z) \neq 0$ be analytic in Δ and

$$Q(z) := zq'(z)q^{\frac{1}{\alpha}-1}(z)$$

be starlike of order β in Δ . If $p(z)$ is analytic in Δ and

$$p(z) \left(1 + \frac{zp'(z)}{p(z)} \right)^\alpha \prec q(z) \left(1 + \frac{zq'(z)}{q(z)} \right)^\alpha, \quad (3.3.1)$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Since the subordination (3.3.1) is equivalent to

$$p^{\frac{1}{\alpha}}(z) + zp'(z)p^{\frac{1}{\alpha}-1}(z) \prec q^{\frac{1}{\alpha}}(z) + zq'(z)q^{\frac{1}{\alpha}-1}(z), \quad (3.3.2)$$

the result follows by an application of Lemma 2.1.

Remark 3.1. Let $0 < \alpha \leq 1$. If $q(z)$ is a convex function that maps Δ onto a region in the right half-plane, then we see that $Q(z)$ is starlike and the hypothesis of Lemma 3.1 is satisfied.

By making use of the above Lemma 3.1, we obtain the following:

Theorem 3.2. Let $\alpha \neq 0$ be any complex number and $\beta = \max\{0, -\Re\frac{1}{\alpha}\}$. Let $q(z) \neq 0$ be analytic in Δ and

$$Q(z) = zq'(z)q^{\frac{1}{\alpha}-1}(z)$$

be starlike of order β in Δ . If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\Phi(f(z))} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{\Phi'(f(z))}{\Phi(f(z))} zf'(z) \right)^\alpha \prec q(z) \left(1 + \frac{zq'(z)}{q(z)} \right)^\alpha, \quad (3.3.3)$$

then

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. By taking

$$p(z) := \frac{zf'(z)}{\Phi(f(z))} \quad (z \in \Delta), \quad (3.3.4)$$

we see that Theorem 3.2 follows from Lemma 3.1.

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