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SOME RESULTS OF OPERATOR IDEALS ON s-TYPE $|A,p|$ OPERATORS

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Abstract. Let $s=(s_n)$ be a sequence of s-numbers in the sense of Pietsch and A be an infinite matrix. This paper presents a generalized class $\mathscr{A}^{(s)}-p$ of s-type $|A,p|$ operators using s-number sequence which unifies many earlier well known classes. It is shown that the class $\mathscr{A}^{(s)}-p$ forms a quasi-Banach operator ideal under certain conditions on the matrix A. Moreover, the inclusion relations among the operator ideals as well as the inclusion relations among their duals are established. It is also proved that for the Cesàro matrix of order 1, the operator ideal formed by approximation numbers is small for $1<p<\infty$.

1. Introduction

Due to the immense applications in spectral theory, the geometry of Banach spaces, theory of eigenvalue distributions, etc., the theory of operator ideals occupies a special importance in functional analysis. Many useful operator ideals have been defined by using sequence of s-numbers. In 1963, Pietsch [4] firstly introduced the approximation numbers of a bounded linear operator in Banach spaces. Subsequently, different s-numbers, namely Kolmogorov numbers, Gel'fand numbers, etc. are introduced to the Banach space setting. For the unifications of different s-numbers, Pietsch ([5], 1974) defined an axiomatic theory of s-numbers in Banach spaces.

For each fixed infinite matrix $A=(a_{nk})$, Rhoades [11] defined $A-p$ space, denoted by $|A,p|$ as

$$|A,p|=\begin{cases} x\in w:\Big(\sum\limits_{n=1}^{\infty}\Big(\sum\limits_{k=1}^{\infty}|a_{nk}x_k|\Big)^p\Big)^{\frac{1}{p}}<\infty & \text{for } 0<p<\infty \\ x\in w:\sup\limits_{n\geq 1}\Big(\sum\limits_{k=1}^{\infty}|a_{nk}x_k|\Big)<\infty & \text{for } p=\infty, \end{cases}$$

where w is a sequence space of real or complex numbers. Further, Rhoades [12] has shown that if $A=(a_{nk})$ is a triangle, i.e., $a_{nk}=0$ for $k>n$ and $a_{nn}\neq 0$, then the space $|A,p|$ is separable for $1<p<\infty$ and complete for $1<p\leq\infty$. $A-p$ spaces contain many known sequence

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spaces such as Cesàro sequence spaces, $1 \le p < \infty$ [13], l_p sequence spaces, $0 < p \le \infty$ etc. by specifying suitable matrix. In particular, if we choose the matrix $A = (a_{nk})$ as a Nörlund matrix, i.e.,

$$a_{nk} = \begin{cases} \frac{a_{n+1-k}}{A_n} : & 1 \le k \le n \\ 0 \quad : & k > n, \end{cases}$$

where a_n is nonnegative for each n and $A_n = \sum_{k=1}^{n} a_k > 0$, then $|A, p|$ space reduces to

$$\Big\{ x \in w : \Big(\sum_{n=1}^{\infty} \Big(\frac{1}{A_n} \sum_{k=1}^{n} |a_{n+1-k} x_k| \Big)^p \Big)^{\frac{1}{p}} < \infty \Big\},$$

whereas Nörlund sequence space [14] is defined as, for $1 \le p < \infty$

$$\Big\{ x \in w : \Big(\sum_{n=1}^{\infty} \Big(\frac{1}{A_n} | \sum_{k=1}^{n} a_{n+1-k} x_k| \Big)^p \Big)^{\frac{1}{p}} < \infty \Big\}.$$

If we choose the sequence $a_n = 1$ for all $n \in \mathbb{N}$, then $|A, p|$ space reduces to Cesàro sequence space for $1 \le p < \infty$, denoted as ces_p. For $A = I$, we have the sequence space l_p for $0 < p \le \infty$.

Pietsch [4] calls an operator $T \in \mathscr{L}(E, F)$ to be l^p type if $\sum_{n=1}^{\infty} (a_n(T))^p$ is finite for $0 < p < \infty$, where $(a_n(T))$ is the sequence of approximation numbers of the bounded linear operator T. Later on Constantin [2] generalized the class of l_p type operators to the class of $ces - p$ type operators by using the Cesàro sequence spaces, where an operator $T \in \mathscr{L}(E, F)$ is called $ces - p$ type if $\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=1}^{n} a_k(T) \Big)^p$ is finite, $1 < p < \infty$. Rhoades [11] further generalized the class of $ces - p$ type operators to the class of $A - p$ type operators, where $A = (a_{nk})$ is an arbitrary infinite matrix. An operator $T \in \mathscr{L}(E, F)$ is said to be $A - p$ type operator if $(a_n(T))$ is an element of the corresponding $|A, p|$ space, $0 < p \le \infty$. Pietsch [7] studied extensively operator ideals generated from the different s-number sequences by using Lorentz sequence spaces.

The purpose of this paper is to study a generalized class of operators using the sequence of s-numbers. We have also shown that the class $\mathscr{A}^{(s)} - p$ of s-type $|A, p|$ operators is a quasi-Banach operator ideal under some certain conditions on the infinite matrix A, which is more general than the usual classes of operator ideals. Moreover, we have obtained various inclusion relations among the operator ideals as well as the inclusion relations among their duals. Finally, it is shown that for the Cesàro matrix of order 1, the operator ideal formed by approximation numbers is small for $1 < p < \infty$.

2. Preliminaries

Throughout this paper we denote E, F as the real or complex Banach spaces and $\mathscr{L}(E, F)$ as the space of all bounded linear operators from E to F. Let $\mathscr{L}$ be the class of all bounded linear operators between arbitrary Banach spaces. We denote $E^{'}$ as the dual of E and $x^{'}$ is the continuous linear functional on E. $\mathbb{N}$ and $\mathbb{R}^{+}$ stand for the set of all natural numbers and the set of all nonnegative real numbers, respectively. Let $x^{'} \in E^{'}$ and $y \in F$, then the map $x^{'} \otimes y : E \to F$ is defined by $(x^{'} \otimes y)(x) = x^{'}(x)y, \ x \in E$.

We now state few results which will be used in the sequel. Before it, we recall some basic definitions and terminologies of s**-numbers of operators** and **operator ideals**.

Definition 2.1. A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 2.2 ([1], [9])**.** A map $s = (s_n) : \mathscr{L} \to \mathbb{R}^{+}$ assigning to every operator $T \in \mathscr{L}$ a non-negative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an s-number sequence if the following conditions are satisfied:

($S1$) monotonicity: $\|T\| = s_1(T) \geq s_2(T) \geq \cdots \geq 0, \quad$ for $T \in \mathscr{L}(E,F)$

($S2$) additivity: $s_{m+n-1}(S+T) \leq s_m(S) + s_n(T), \quad$ for $S, T \in \mathscr{L}(E,F)$, $m, n \in \mathbb{N}$

($S3$) ideal property: $s_n(RST) \leq \|R\| s_n(S) \|T\|, \quad$ for some $R \in \mathscr{L}(F, F_0)$, $S \in \mathscr{L}(E,F)$ and $T \in \mathscr{L}(E_0, E)$, where E_0, F_0 are arbitrary Banach spaces

($S4$) rank property: If $rank(T) \leq n$ then $s_n(T) = 0$

($S5$) norming property: $s_n(I : l_2^n \to l_2^n) = 1$, where I denotes the identity operator on the n-dimensional Hilbert space l_2^n.

We call $s_n(T)$ the n-th s-number of the operator T. For results on s-number sequence, refer ([1], [5], [7], [8], [9]).
We give some examples of s-number sequences of a bounded linear operator.
Let $T \in \mathscr{L}(E,F)$ and $n \in \mathbb{N}$.
The n-th approximation number, denoted by $a_n(T)$, is defined as

$$a_n(T) = \inf\Big\{\|T - L\| : \quad L \in \mathscr{L}(E,F), \ \text{rank(L)} < \text{n}\Big\}.$$

The n-th Gel'fand number, denoted by $c_n(T)$, is defined as

$$c_n(T) = \inf\Big\{\|TJ_M\| : \quad M \subset E, \ \text{codim(M)} < \text{n}\Big\},$$

where $J_M : M \to E$ be the natural embedding from subspace M of E into E.
The n-th Kolmogorov number, denoted by $d_n(T)$, is defined as

$$d_n(T) = \inf\Big\{\|Q_N(T)\| : \quad N \subset F, \ \text{dim(N)} < \text{n}\Big\},$$

where $Q_N : E \to E/N$ be the quotient map from E onto E/N.
The n-th Weyl number, denoted by $x_n(T)$, is defined as

$$x_n(T) = \inf\left\{a_n(TA) : \|A : l_2 \to E\| \le 1\right\},$$

where $a_n(TA)$ is an n-th approximation number of the operator TA.
The n-th Chang number, denoted by $y_n(T)$, is defined as

$$y_n(T) = \inf\left\{a_n(BT) : \|B : F \to l_2\| \le 1\right\},$$

where $a_n(BT)$ is an n-th approximation number of the operator BT.
The n-th Hilbert number, denoted by $h_n(T)$, is defined as

$$h_n(T) = \sup\left\{a_n(BTA) : \|B : F \to l_2\| \le 1,\ \|A : l_2 \to E\| \le 1\right\}.$$

Remark 2.1 ([9])**.** Among all the s-number sequences defined above, it is easy to verify that the approximation number, $a_n(T)$ is the largest and the Hilbert number, $h_n(T)$ is the smallest s-number sequence, i.e., $h_n(T) \le s_n(T) \le a_n(T)$ for any bounded linear operator T. If T is compact and defined on a Hilbert space, then all the s-numbers coincide with the singular values of T, i.e., the eigenvalues of $|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$.

Proposition 2.1 ([9], p.115)**.** *Let $T \in \mathscr{L}(E,F)$. Then*
$h_n(T) \le x_n(T) \le c_n(T) \le a_n(T)$ *and* $h_n(T) \le y_n(T) \le d_n(T) \le a_n(T)$.

Definition 2.3. ([9], p.90) An s-number sequence $s = (s_n)$ is called injective if, given any metric injection $J \in \mathscr{L}(F, F_0)$, $s_n(T) = s_n(JT)$ for all $T \in \mathscr{L}(E,F)$.

Definition 2.4. ([9], p.95) An s-number sequence $s = (s_n)$ is called surjective if, given any metric surjection $Q \in \mathscr{L}(E_0, E)$, $s_n(T) = s_n(TQ)$ for all $T \in \mathscr{L}(E,F)$.

Proposition 2.2. ([9], pp.90-94) *The Gel'fand numbers and the Weyl numbers are injective.*

Proposition 2.3. ([9], p.95) *The Kolmogorov numbers and the Chang numbers are surjective.*

The following lemma is required to prove our theorems.

Lemma 2.1 ([5])**.** $|s_n(T) - s_n(S)| \le \|T - S\|$ for $S, T \in \mathscr{L}(E,F)$ and $n = 1, 2, \cdots$.

Definition 2.5 ((**Dual s-numbers**) [5])**.** For each s-number sequence $s = (s_n)$, a dual s-number function $s^D = (s_n^D)$ is defined by

$$s_n^D(T) = s_n(T') \quad \text{for all } T \in \mathscr{L},$$

where T' is the dual of T.

Definition 2.6 ([7], p.152)**.** An s-number sequence is called symmetric if $s_n(T) \geq s_n(T^{'})$ for all $T \in \mathscr{L}$. If $s_n(T) = s_n(T^{'})$ then the s-number sequence is said to be completely symmetric.

Now we state some known results of dual of an s-number sequence.

Theorem 2.1. ([7], p.152) *The approximation numbers are symmetric, i.e.,* $a_n(T^{'}) \leq a_n(T)$ *for* $T \in \mathscr{L}$.

Remark 2.2. $a_n(T^{'}) = a_n(T)$ for every compact operator T (refer, C. V. Hutton [3]).

Theorem 2.2 ([7], p.153)**.** *Let* $T \in \mathscr{L}$. *Then*

$$c_n(T) = d_n(T^{'}) \quad and \quad c_n(T^{'}) \leq d_n(T).$$

In addition, if T *is a compact operator then* $c_n(T^{'}) = d_n(T)$.

Theorem 2.3. ([9], p.96) *Let* $T \in \mathscr{L}$. *Then*

$$x_n(T) = y_n(T^{'}) \quad and \quad y_n(T) = x_n(T^{'}),$$

i.e., Weyl numbers and Chang numbers are dual to each other.

Theorem 2.4 ([7], p.153)**.** *The Hilbert numbers are completely symmetric, i.e.,* $h_n(T) = h_n(T^{'})$ *for all* $T \in \mathscr{L}$.

Definition 2.7 ([7], [10])**.** Let $\mathscr{L}$ be the class of all bounded linear operators between arbitrary Banach spaces and $\mathscr{L}(E,F)$ be the set of all such operators from E to F. A sub collection $\mathscr{M}$ of $\mathscr{L}$ is said to be an ideal if each component $\mathscr{M}(E,F) = \mathscr{M} \bigcap \mathscr{L}(E,F)$ satisfies the following conditions:

($OI1$) if $x' \in E'$, $y \in F$ then $x' \otimes y \in \mathscr{M}(E,F)$;

($OI2$) if $S, T \in \mathscr{M}(E,F)$ then $S + T \in \mathscr{M}(E,F)$;

($OI3$) if $S \in \mathscr{M}(E,F)$, $T \in \mathscr{L}(E_0,E)$ and $R \in \mathscr{L}(F,F_0)$ then $RST \in \mathscr{M}(E_0,F_0)$.

Definition 2.8 ([7], [10])**.** A function $\alpha : \mathscr{M} \to \mathbb{R}^+$ is said to be a quasi-norm on the ideal $\mathscr{M}$ if the following conditions hold:

($QON1$) if $x' \in E'$, $y \in F$ then $\alpha(x' \otimes y) = \|x^{'}\| \|y\|$;

($QON2$) there exists a constant $C_\alpha \geq 1$ such that $\alpha(S+T) \leq C_\alpha[\alpha(S)+\alpha(T)]$ for $S, T \in \mathscr{M}(E,F)$;

($QON3$) if $S \in \mathscr{M}(E,F)$, $T \in \mathscr{L}(E_0,E)$ and $R \in \mathscr{L}(F,F_0)$ then $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$.

In particular if $C_\alpha = 1$ then α becomes a norm on the operator ideal $\mathscr{M}$.

An ideal $\mathscr{M}$ with a quasi-norm α, denoted by $[\mathscr{M}, \alpha]$ is said to be a quasi-Banach operator ideal if each component $\mathscr{M}(E,F)$ is complete under the quasi-norm α. A quasi-normed operator ideal $[\mathscr{M}, \alpha]$ is called injective if for every operator $T \in \mathscr{L}(E,F)$ and a metric injection $J \in \mathscr{L}(F, F_0)$, $JT \in \mathscr{M}(E, F_0)$ we have $T \in \mathscr{M}(E,F)$ and $\alpha(JT) = \alpha(T)$. Further, a quasi-normed operator ideal $[\mathscr{M}, \alpha]$ is called surjective if for every operator $T \in \mathscr{L}(E,F)$ and a metric surjection $Q \in \mathscr{L}(E_0, E)$, $TQ \in \mathscr{M}(E_0, F)$ we have $T \in \mathscr{M}(E,F)$ and $\alpha(TQ) = \alpha(T)$. Thus injectivity and surjectivity are dual concept. For its various properties, please refer to [7].

Definition 2.9 ([7], [10])**.** For every operator ideal $\mathscr{M}$, the dual operator ideal denoted by $\mathscr{M}'$ is defined as

$$\mathscr{M}'(E,F) = \left\{T \in \mathscr{L}(E,F) : \quad T' \in \mathscr{M}(F', E')\right\},$$

where T' is the dual of T and E' and F' are the duals of E and F, respectively.

Definition 2.10 ([7], p.68)**.** An operator ideal $\mathscr{M}$ is called symmetric if $\mathscr{M} \subset \mathscr{M}'$ and is called completely symmetric if $\mathscr{M} = \mathscr{M}'$.

Remark 2.3 ([7], [10])**.**

1. $\mathscr{M}'$ is complete if $\mathscr{M}$ is complete.
2. If $[\mathscr{M}', \alpha']$ be the dual of quasi-normed ideal $[\mathscr{M}, \alpha]$ then $\alpha'(T) = \alpha(T')$.

We now consider some known operator ideals determined by sequence of s-numbers, namely $\mathscr{L}^{(s)}_{r,p}$ and $S^{(s)}_p$ (see [7], [9]), where

$$\mathscr{L}^{(s)}_{r,p} := \left\{T \in \mathscr{L} : \sum_{n=1}^{\infty} \left(n^{\frac{1}{r}-\frac{1}{p}} s_n(T)\right)^p < \infty\right\} \quad \text{for } 0 < r, p < \infty,$$

and

$$S^{(s)}_p := \left\{T \in \mathscr{L} : \sum_{n=1}^{\infty} \left(s_n(T)\right)^p < \infty\right\} \qquad \text{for } 0 < p < \infty.$$

Definition 2.11 ([6])**.** An operator ideal $\mathscr{M}$ is said to be small if $\mathscr{M}(E,F) = \mathscr{L}(E,F)$ implies that at least one of the Banach spaces E and F is of finite dimension.

3. Operators of s-type $|A, p|$

In this section we have defined s-type $|A, p|$ operators and proved that the operator ideal formed by s-number sequence is complete. We have also investigated the dual and inclusion results among the operator ideals.

Let $s=(s_n)$ be a sequence of s-numbers. We call an operator $T\in\mathscr{L}(E,F)$ is of s-type $|A,p|$ operator if the sequence of s-numbers of T, $(s_n(T))$ is an element of the corresponding $|A,p|$ space. In other words, T is of s-type $|A,p|$ operator if

$$\begin{cases}\Big(\sum\limits_{n=1}^{\infty}\Big(\sum\limits_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)^p\Big)^{\frac{1}{p}}<\infty & \text{for}\quad 0<p<\infty,\\ \sup\limits_{n\geq 1}\Big(\sum\limits_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)<\infty & \text{for}\quad p=\infty.\end{cases}$$

For each fixed matrix A, we denote $\mathscr{A}^{(s)}-p$ be the class of all s-type $|A,p|$ operators between arbitrary Banach spaces and $\mathscr{A}^{(s)}_{(E\to F)}-p$ be the set of s-type $|A,p|$ operators from E to F which is a component of $\mathscr{A}^{(s)}-p$ for $0<p\leq\infty$.

Let $A=(a_{nk})$ be a matrix satisfying the condition:

$$|a_{n,2k-1}|+|a_{n,2k}|\leq M|a_{nk}|\qquad\text{for each } k \text{ and } n, \tag{3.1}$$

where M is a constant independent of n and k.

Theorem 3.1. *Let $0<p<\infty$. For fixed matrix $A=(a_{nk})$ satisfying (3.1) and $\sum\limits_{n=1}^{\infty}|a_{n1}|^p<\infty$, the class $\mathscr{A}^{(s)}-p$ is an operator ideal.*

Proof. Let E and F be any two Banach spaces. We shall prove $(OI1)$ to $(OI3)$ to show $\mathscr{A}^{(s)}-p$ is an operator ideal. Let $x'\in E'$, $y\in F$ then $x'\otimes y$ is a rank one operator. So

$$s_n(x'\otimes y)=0\quad\text{ for all } n\geq 2.$$

We have

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(x'\otimes y)|\Big)^p\Big)^{\frac{1}{p}}&=\Big(\sum_{n=1}^{\infty}\Big(|a_{n1}s_1(x'\otimes y)|\Big)^p\Big)^{\frac{1}{p}}\\ &=\|x'\otimes y\|\Big(\sum_{n=1}^{\infty}|a_{n1}|^p\Big)^{\frac{1}{p}}<\infty.\end{aligned}$$

Thus $x'\otimes y\in\mathscr{A}^{(s)}_{(E\to F)}-p$ and hence $(OI1)$ is proved.

Let $S,T\in\mathscr{A}^{(s)}_{(E\to F)}-p$. We calculate

$$\begin{aligned}\sum_{k=1}^{\infty}|a_{nk}s_k(T+S)|&=\sum_{k=1}^{\infty}|a_{n,2k-1}s_{2k-1}(T+S)|+\sum_{k=1}^{\infty}|a_{n,2k}s_{2k}(T+S)|\\ &\leq\Big(\sum_{k=1}^{\infty}\Big(|a_{n,2k-1}|+|a_{n,2k}|\Big)s_{2k-1}(T+S)\Big)\end{aligned}$$

$$\leq M\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(T)+\sum_{k=1}^{\infty}|a_{nk}|s_k(S)\Big). \tag{3.2}$$

Case I: $0<p<1$

For $0<p<1$ and $a,b>0$, we have $\Big(a+b\Big)^p\leq\Big(a^p+b^p\Big)$ and $\Big(a+b\Big)^{\frac{1}{p}}\leq C\Big(a^{\frac{1}{p}}+b^{\frac{1}{p}}\Big)$, where $C\geq 1$ is a constant.

From (3.2), we have

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T+S)|\Big)^p\Big)^{\frac{1}{p}} &\leq M\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(T)+\sum_{k=1}^{\infty}|a_{nk}|s_k(S)\Big)^p\Big)^{\frac{1}{p}}\\ &\leq C.M\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(T)\Big)^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(S)\Big)^p\Big)^{\frac{1}{p}}\Big] \quad <\infty\end{aligned}$$

where $C\geq 1$ is a constant.

Case II: $1\leq p<\infty$

Using Minkowski inequality for $1\leq p<\infty$, we have from (3.2)

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T+S)|\Big)^p\Big)^{\frac{1}{p}} &\leq M\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(T)+\sum_{k=1}^{\infty}|a_{nk}|s_k(S)\Big)^p\Big)^{\frac{1}{p}}\\ &\leq M\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(T)\Big)^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}|s_k(S)\Big)^p\Big)^{\frac{1}{p}}\Big] \quad <\infty.\end{aligned}$$

Thus $S+T\in\mathscr{A}^{(s)}_{(E\to F)}-p$ and hence $(OI2)$ is proved.

Let $T\in\mathscr{L}(E_0,E)$, $R\in\mathscr{L}(F,F_0)$ and $S\in\mathscr{A}^{(s)}_{(E\to F)}-p$. It is required to prove $RST\in\mathscr{A}^{(s)}_{(E_0\to F_0)}-p$. Using the property $(S3)$ in the Definition 2.2., we have

$$s_n(RST)\leq\|R\|s_n(S)\|T\| \quad \text{for all } n\in\mathbb{N}.$$

So

$$\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(RST)|\Big)^p\Big)^{\frac{1}{p}}\leq\|R\|\|T\|\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(S)|\Big)^p\Big)^{\frac{1}{p}} <\infty.$$

Thus $RST\in\mathscr{A}^{(s)}_{(E_0\to F_0)}-p$ and therefore $(OI3)$ is proved.

Hence $\mathscr{A}^{(s)}-p$ is an operator ideal. □

Corollary 3.1. *Let $A=(a_{nk})$ be a matrix satisfying (3.1) and $\sup_{n\geq 1}|a_{n1}|<\infty$. Then for $p=\infty$, the class $\mathscr{A}^{(s)}-\infty$ is an operator ideal.*

Remark 3.1. Let $A=(a_{nk})$ be a matrix such that $\sum_{n=1}^{\infty}|a_{n1}|^p<\infty$, then the condition (3.1) on the matrix is sufficient but not necessary to form an operator ideal. Justification is given below.

Justification: Let $0<p<\infty$ and $A=(a_{nk})$ be a nonzero diagonal matrix such that

$$|a_{2n-1,2n-1}|^p+|a_{2n,2n}|^p\le M_1|a_{nn}|^p, \tag{3.3}$$

where M_1 is a constant independent of n.
First of all, we show that for the nonzero diagonal matrix A satisfying (3.3), the class $\mathscr{A}^{(s)}-p$ of s-type $|A,p|$ operators forms an operator ideal. For this let $S,T\in\mathscr{A}^{(s)}_{(E\to F)}-p$. Then

$$\begin{aligned}\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T+S)|\Big)^p\Big)^{\frac{1}{p}} &= \Big(\sum_{n=1}^{\infty}\Big(|a_{nn}|s_n(T+S)\Big)^p\Big)^{\frac{1}{p}}\\ &\le C_1.M_1^{\frac{1}{p}}\Big[\Big(\sum_{n=1}^{\infty}\Big(|a_{nn}|s_n(T)\Big)^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{\infty}\Big(|a_{nn}|s_n(S)\Big)^p\Big)^{\frac{1}{p}}\Big]\\ &= C_1.M_1^{\frac{1}{p}}\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)^p\Big)^{\frac{1}{p}}+\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(S)|\Big)^p\Big)^{\frac{1}{p}}\Big]\\ &<\infty.\end{aligned}$$

Thus $S+T\in\mathscr{A}^{(s)}_{(E\to F)}-p$. Hence $(OI2)$ is proved. Clearly the conditions $(OI1)$ and $(OI3)$ hold good. Thus the class $\mathscr{A}^{(s)}-p$ is an operator ideal. But the nonzero diagonal matrix A (In particular identity matrix) does not satisfy the condition (3.1). This proves our claim.

Remark 3.2. Let $A=I$, an identity matrix, then the operator ideal $\mathscr{A}^{(s)}-p$ becomes a well known operator ideal $S_p^{(s)}$ which has been studied extensively by many mathematicians.
If we choose the matrix $A=(a_{nk})$ such that for $0<r,p<\infty$

$$a_{nk}=\begin{cases} n^{\frac{1}{r}-\frac{1}{p}} : & \text{if } n=k\\ 0 \quad : & \text{otherwise.}\end{cases}$$

Then the matrix A satisfies the condition (3.3) and forms a quasi-Banach operator ideals denoted as $\mathscr{L}^{(s)}_{r,p}$ introduced by Pietsch.

Note 3.1. It is observed that if the matrix $A=(a_{nk})$ satisfies the condition (3.1), then the set $\mathscr{A}^{(s)}_{(E\to F)}-p$ of s-type $|A,p|$ operators from E to F is a linear space. So if we choose the s-number sequence as the sequence of approximation numbers, then the set $\mathscr{A}^{(a)}_{(E\to F)}-p$ is same as the set of $A-p$ type operators studied by Rhoades. If we choose the matrix A as the Cesàro matrix of order 1 then the set $\mathscr{A}^{(a)}_{(E\to F)}-p$ coincides with the set of $ces-p$ type operators introduced by Constantin.

Note 3.2. Rhoades [11] raised an open question whether the condition (3.1) on the matrix $A=(a_{nk})$ is necessary for the set of $A-p$ type operator to form a linear space? It is observed that the identity matrix I does not satisfy the condition (3.1) but the set of $A-p$ type (here $A=I$) operators forms a linear space structure. This answers the question of Rhoades in negation, i.e., the condition (3.1) on the matrix A is sufficient but not a necessary to form a linear space.

Proposition 3.1. *For $1\le p<q\le\infty$, we have $\mathscr{A}^{(s)}-p\subseteq\mathscr{A}^{(s)}-q$.*

Proof. For $1\le p<q\le\infty$, we have $|A,p|\subseteq|A,q|$. So the proof of Proposition 3.1. is trivial. □

Let $\mathscr{A}^{(s)}-p$ be an operator ideal. Define $\beta^{(s)}_{A,p}:\mathscr{A}^{(s)}-p\to\mathbb{R}^{+}$ for $0<p<\infty$ by

$$\beta^{(s)}_{A,p}(T)=\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)^{p}\Big)^{\frac{1}{p}},$$

where $T\in\mathscr{A}^{(s)}-p$.

Note 3.3. For $p=\infty$, we define $\beta^{(s)}_{A,\infty}(T)=\sup\limits_{n\ge1}\Big(\sum\limits_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)$.

Theorem 3.2. *Let $0<p<\infty$. For fixed nonzero matrix $A=(a_{nk})$ satisfying the condition (3.1) and $\sum\limits_{n=1}^{\infty}|a_{n1}|^p<\infty$, the function $\hat{\beta}^{(s)}_{A,p}$ is a quasi-norm on the operator ideal $\mathscr{A}^{(s)}-p$, where* $\hat{\beta}^{(s)}_{A,p}(T)=\dfrac{\beta^{(s)}_{A,p}(T)}{\Big(\sum\limits_{n=1}^{\infty}|a_{n1}|^p\Big)^{\frac{1}{p}}},\quad T\in\mathscr{A}^{(s)}-p.$

Proof. Let E and F be two Banach spaces and $\mathscr{A}^{(s)}_{(E\to F)}-p$ be any one of the components of $\mathscr{A}^{(s)}-p$.
Let $x'\in E'$, $y\in F$, then $x'\otimes y$ is a rank one operator. So $s_n(x'\otimes y)=0$, $\forall n\ge2$. Therefore,

$$\beta^{(s)}_{A,p}(x'\otimes y)=\Big(\sum_{n=1}^{\infty}\Big(|a_{n1}s_1(x'\otimes y)|\Big)^{p}\Big)^{\frac{1}{p}}=\|x'\otimes y\|\Big(\sum_{n=1}^{\infty}|a_{n1}|^{p}\Big)^{\frac{1}{p}}.$$

Again $\|x'\otimes y\|=\sup\limits_{\|x\|=1}\|(x'\otimes y)(x)\|=(\sup\limits_{\|x\|=1}|x'(x)|)\|y\|=\|x'\|\|y\|$.
Therefore

$$\hat{\beta}^{(s)}_{A,p}(x'\otimes y)=\|x'\|\|y\|.$$

Suppose that $S,T\in\mathscr{A}^{(s)}_{(E\to F)}-p$, then

$$\beta^{(s)}_{A,p}(S+T)=\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(S+T)|\Big)^{p}\Big)^{\frac{1}{p}}$$

$$
\begin{aligned}
&= \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{n,2k-1}s_{2k-1}(S+T)| + \sum_{k=1}^{\infty}|a_{n,2k}s_{2k}(S+T)|\Big)^p\Big)^{\frac{1}{p}}\\
&\le C_\beta.M\Big(\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(S)|\Big)^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)^p\Big)^{\frac{1}{p}}\Big)\\
&\le C_\beta.M\Big(\beta_{A,p}^{(s)}(S) + \beta_{A,p}^{(s)}(T)\Big),
\end{aligned}
$$

where $C_\beta \ge 1$ is a constant. Thus

$$\hat{\beta}_{A,p}^{(s)}(S+T) \le C_\beta.M\Big(\hat{\beta}_{A,p}^{(s)}(S) + \hat{\beta}_{A,p}^{(s)}(T)\Big).$$

Finally, let $S \in \mathscr{A}_{(E\to F)}^{(s)} - p$, $R \in \mathscr{L}(F,F_0)$ and $T \in \mathscr{L}(E_0,E)$. Then

$$
\begin{aligned}
\beta_{A,p}^{(s)}(RST) &= \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(RST)|\Big)^p\Big)^{\frac{1}{p}}\\
&\le \|R\|\|T\|\Big(\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(S)|\Big)^p\Big)^{\frac{1}{p}}\Big)\\
&\le \|R\|\beta_{A,p}^{(s)}(S)\|T\|.
\end{aligned}
$$

Thus

$$\hat{\beta}_{A,p}^{(s)}(RST) \le \|R\|\hat{\beta}_{A,p}^{(s)}(S)\|T\|.$$

Hence $\hat{\beta}_{A,p}^{(s)}$ is a quasi-norm on the operator ideal $\mathscr{A}^{(s)} - p$. □

Example 3.1. Let $A = (a_{nk})$ be a Cesàro matrix of order 1, then for $p = 2$, $\hat{\beta}_{A,2}^{(s)}$ is a quasi norm on the operator ideal $\mathscr{A}^{(s)} - 2$, where $\hat{\beta}_{A,2}^{(s)}(T) = \frac{\sqrt{6}}{\pi}\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}s_k(T)\Big)^2\Big)^{\frac{1}{2}}$, $T \in \mathscr{A}^{(s)} - 2$.

Corollary 3.2. *Let $A = (a_{nk})$ be a nonzero matrix satisfying (3.1) and $\sup_{n\ge1}|a_{n1}| < \infty$. Then for $p = \infty$, the function $\hat{\beta}_{A,\infty}^{(s)}$ is a quasi-norm on the operator ideal $\mathscr{A}^{(s)} - \infty$, where $\hat{\beta}_{A,\infty}^{(s)}(T) = \frac{\beta_{A,\infty}^{(s)}(T)}{\sup_{n\ge1}|a_{n1}|}$, $T \in \mathscr{A}^{(s)} - \infty$.*

Theorem 3.3. *The operator ideal $\mathscr{A}^{(s)} - p$ is complete with the quasi-norm $\hat{\beta}_{A,p}^{(s)}$, i.e., $[\mathscr{A}^{(s)} - p, \hat{\beta}_{A,p}^{(s)}]$ is a quasi-Banach operator ideal for $0 < p \le \infty$.*

Proof. Let $0 < p < \infty$. To prove $\mathscr{A}^{(s)} - p$ is a quasi-Banach operator ideal, it is enough to prove that each component $\mathscr{A}_{(E\to F)}^{(s)} - p$ of $\mathscr{A}^{(s)} - p$ is complete under the quasi norm $\hat{\beta}_{A,p}^{(s)}$.
We have

$$\beta_{A,p}^{(s)}(T) = \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)^p\Big)^{\frac{1}{p}}$$

$$\geq \left(\sum_{n=1}^{\infty}\left(|a_{n1}s_1(T)|\right)^p\right)^{\frac{1}{p}}$$
$$= \|T\|\left(\sum_{n=1}^{\infty}|a_{n1}|^p\right)^{\frac{1}{p}}.$$

$$\Rightarrow \|T\| \leq \hat{\beta}^{(s)}_{A,p}(T) \quad \text{for } T \in \mathscr{A}^{(s)}_{(E\to F)} - p. \tag{3.4}$$

Let (T_m) be a Cauchy sequence in $\mathscr{A}^{(s)}_{(E\to F)} - p$. Then $\forall\ \epsilon > 0$, there exist $N \in \mathbb{N}$ such that

$$\hat{\beta}^{(s)}_{A,p}(T_m - T_l) < \epsilon, \quad \forall\ m, l \geq N. \tag{3.5}$$

Now from (3.4),

$$\|T_m - T_l\| \leq \hat{\beta}^{(s)}_{A,p}(T_m - T_l).$$

Using (3.5), we have

$$\|T_m - T_l\| \leq \hat{\beta}^{(s)}_{A,p}(T_m - T_l) < \epsilon \quad \forall\ m, l \geq N.$$

Hence (T_m) is a Cauchy sequence in $\mathscr{L}(E,F)$. As F is a Banach space, $\mathscr{L}(E,F)$ is also a Banach space. Therefore $T_m \to T$ as $m \to \infty$ in $\mathscr{L}(E,F)$. We shall now show that $T_m \to T$ as $m \to \infty$ in $\mathscr{A}^{(s)}_{(E\to F)} - p$.

Using Lemma 2.1., we have

$$|s_n(T_l - T_m) - s_n(T - T_m)| \leq \|T_l - T\|.$$

Letting $l \to \infty$, we have

$$s_n(T_l - T_m) \to s_n(T - T_m). \tag{3.6}$$

From (3.5), we get

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{nk}s_k(T_l - T_m)|\right)^p\right)^{\frac{1}{p}} < \epsilon\left(\sum_{n=1}^{\infty}|a_{n1}|^p\right)^{\frac{1}{p}}, \quad \forall\ m, l \geq N.$$

Using (3.6), it can be shown that as $l \to \infty$ (keeping $m \geq N$ fixed)

$$\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{nk}s_k(T - T_m)|\right)^p\right)^{\frac{1}{p}} \leq \epsilon\left(\sum_{n=1}^{\infty}|a_{n1}|^p\right)^{\frac{1}{p}}$$
$$\Rightarrow \hat{\beta}^{(s)}_{A,p}(T - T_m) \leq \epsilon \quad \forall\ m \geq N.$$

This means that $T_m \to T$ under the quasi-norm $\hat{\beta}^{(s)}_{A,p}$.

Next to show that $T \in \mathscr{A}^{(s)}_{(E\to F)} - p$. Now

$$\sum_{k=1}^{\infty}|a_{nk}s_k(T)| = \sum_{k=1}^{\infty}|a_{n,2k-1}s_{2k-1}(T)| + \sum_{k=1}^{\infty}|a_{n,2k}s_{2k}(T)|$$

$$\leq \sum_{k=1}^{\infty} \Big(|a_{n,2k-1}| + |a_{n,2k}|\Big) s_{2k-1}(T)$$

Since $0 \leq s_{n+1}(T) \leq s_n(T)$, $\forall\ n$ and using the inequality (3.1), we have

$$\sum_{k=1}^{\infty} |a_{nk} s_k(T)| \leq M\Big(\sum_{k=1}^{\infty} |a_{nk}| s_k(T - T_N) + \sum_{k=1}^{\infty} |a_{nk}| s_k(T_N)\Big).$$

Therefore

$$\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{nk} s_k(T)|\Big)^p\Big)^{\frac{1}{p}} \leq C.M\Big[\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{nk} s_k(T - T_N)|\Big)^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{nk} s_k(T_N)|\Big)^p\Big)^{\frac{1}{p}}\Big]$$
$$< \infty,$$

since $\hat{\beta}^{(s)}_{A,p}(T - T_N) < \infty$ and $(T_N) \in \mathscr{A}^{(s)}_{(E\to F)} - p$.
Hence $T \in \mathscr{A}^{(s)}_{(E\to F)} - p$.

For $p = \infty$, we can similarly prove that $\mathscr{A}^{(s)} - \infty$ is complete under the quasi-norm $\hat{\beta}^{(s)}_{A,p} - \infty$.
This completes the proof. □

We now study some properties of the quasi-Banach operator ideal $\mathscr{A}^{(s)} - p$ for $0 < p \leq \infty$.

Theorem 3.4. *If the s-number sequence is injective, then the quasi-Banach operator ideal $[\mathscr{A}^{(s)} - p, \hat{\beta}^{(s)}_{A,p}]$ is injective for $0 < p \leq \infty$.*

Proof. Let $0 < p < \infty$. Let $T \in \mathscr{L}(E,F)$ and $J \in \mathscr{L}(F,F_0)$ be any metric injection. Suppose that $JT \in \mathscr{A}^{(s)}_{(E\to F_0)} - p$. Then

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{nk} s_k(JT)|\Big)^p < \infty.$$

Since the s-number sequence $s = (s_n)$ is injective, we have $s_n(T) = s_n(JT)$, for all $T \in \mathscr{L}(E,F)$, $n = 1,2,\cdots$. Hence

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{nk} s_k(T)|\Big)^p = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{nk} s_k(JT)|\Big)^p < \infty.$$

Thus $T \in \mathscr{A}^{(s)}_{(E\to F)} - p$ and clearly $\hat{\beta}^{(s)}_{A,p}(JT) = \hat{\beta}^{(s)}_{A,p}(T)$ holds.
Similarly we can prove for $p = \infty$. Hence the operator ideal $[\mathscr{A}^{(s)} - p, \hat{\beta}^{(s)}_{A,p}]$ is injective. □

Remark 3.3. The quasi-Banach operator ideal $[\mathscr{A}^{(c)} - p, \hat{\beta}^{(c)}_{A,p}]$ formed by Gel'fand numbers $c = (c_n)$ and the quasi-Banach operator ideal $[\mathscr{A}^{(x)} - p, \hat{\beta}^{(x)}_{A,p}]$ formed by Weyl numbers $x = (x_n)$ are injective quasi-Banach operator ideals for $0 < p \leq \infty$.

Theorem 3.5. *If the s-number sequence is surjective, then the quasi-Banach operator ideal $[\mathscr{A}^{(s)} - p, \hat{\beta}^{(s)}_{A,p}]$ is surjective for $0 < p \leq \infty$.*

Proof. Let $0 < p < \infty$. Let $T \in \mathscr{L}(E,F)$ and $Q \in \mathscr{L}(E_0,E)$ be any metric surjection. Suppose that $TQ \in \mathscr{A}^{(s)}_{(E_0\to F)} - p$. Then

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(TQ)|\Big)^p < \infty.$$

Since the s-number sequence $s=(s_n)$ is surjective, we have $s_n(T)=s_n(TQ)$, for all $T \in \mathscr{L}(E,F)$ and $n = 1,2,\cdots$. Hence

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(T)|\Big)^p = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}s_k(TQ)|\Big)^p < \infty.$$

Thus $T \in \mathscr{A}^{(s)}_{(E\to F)} - p$ and also $\hat{\beta}^{(s)}_{A,p}(TQ) = \hat{\beta}^{(s)}_{A,p}(T)$.
It is easy to check for $p=\infty$. Hence the operator ideal $[\mathscr{A}^{(s)} - p, \hat{\beta}^{(s)}_{A,p}]$ is surjective. □

Remark 3.4. The quasi-Banach operator ideal $[\mathscr{A}^{(d)} - p, \hat{\beta}^{(d)}_{A,p}]$ formed by Kolmogorov numbers $d=(d_n)$ and the quasi-Banach operator ideal $[\mathscr{A}^{(y)} - p, \hat{\beta}^{(y)}_{A,p}]$ formed by Chang numbers $y=(y_n)$ are surjective quasi-Banach operator ideals.

Let us consider $[\mathscr{A}^{(a)} - p, \hat{\beta}^{(a)}_{A,p}]$ and $[\mathscr{A}^{(h)} - p, \hat{\beta}^{(h)}_{A,p}]$ be the quasi-Banach operator ideals corresponding to the approximation numbers $a=(a_n)$ and the Hilbert numbers $h=(h_n)$ respectively. Then we have the following inclusion relations among the operator ideals.

Theorem 3.6. *Let* $0 < p \le \infty$. *Then*

(I) $\mathscr{A}^{(a)} - p \subseteq \mathscr{A}^{(c)} - p \subseteq \mathscr{A}^{(x)} - p \subseteq \mathscr{A}^{(h)} - p$ and
(II) $\mathscr{A}^{(a)} - p \subseteq \mathscr{A}^{(d)} - p \subseteq \mathscr{A}^{(y)} - p \subseteq \mathscr{A}^{(h)} - p.$

Proof. Let $0 < p < \infty$. Suppose that $T \in \mathscr{A}^{(a)} - p$. Then

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}a_k(T)|\Big)^p < \infty.$$

From Proposition 2.1., we have

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}h_k(T)|\Big)^p \le \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}x_k(T)|\Big)^p \le \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}c_k(T)|\Big)^p \le \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{nk}a_k(T)|\Big)^p.$$

Hence the proof of (I) follows for $0 < p < \infty$. It is trivial to check for $p=\infty$.

We omit the proof of (II) as it is similar to the previous one. □

There are some converse estimates among s-number sequences as given below.

Lemma 3.2 ([8], p.165)**.** *Let* $T \in \mathscr{L}(E,F)$. *Then* $a_n(T) \le 2n^{\frac{1}{2}}c_n(T)$ *and* $a_n(T) \le 2n^{\frac{1}{2}}d_n(T)$.

We have next result related to this converse estimates.

Theorem 3.7. *Let* $0 < r, p < \infty$ *and* $A = (a_{nk})$ *be a diagonal matrix where*

$$a_{nk} = \begin{cases} n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}} : & k = n \\ 0 \quad : & k \neq n. \end{cases}$$

If a bounded linear operator T *from* E *to* F *belongs to* $\mathscr{L}^{(c)}_{r,p}$, *then* T *belongs to* $\mathscr{A}^{(a)} - p$.

Proof. For $0 < p < \infty$, we have

$$\begin{aligned} \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}|a_{nk}a_k(T)|\right)^p &= \sum_{n=1}^{\infty}\left(n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}}a_n(T)\right)^p \\ &\leq \sum_{n=1}^{\infty}\left(n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}}.2n^{\frac{1}{2}}c_n(T)\right)^p \qquad \text{(Using Lemma 3.1.)} \\ &= 2^p\sum_{n=1}^{\infty}\left(n^{\frac{1}{r}-\frac{1}{p}}c_n(T)\right)^p < \infty. \end{aligned}$$

Hence the result follows. □

Remark 3.5. In particular, if we take the diagonal matrix $A = (a_{nk})$, where

$$a_{nk} = \begin{cases} n^{-\frac{1}{2}} : & k = n \\ 0 \quad : & k \neq n. \end{cases}$$

If a bounded linear operator T from E to F belongs to $S^{(c)}_p$, then T belongs to $\mathscr{A}^{(a)} - p$ for $0 < p < \infty$.

Theorem 3.8. *Let* $0 < r, p < \infty$ *and* $A = (a_{nk})$ *be a diagonal matrix where*

$$a_{nk} = \begin{cases} n^{\frac{1}{r}-\frac{1}{p}-\frac{1}{2}} : & k = n \\ 0 \quad : & k \neq n. \end{cases}$$

If a bounded linear operator T *from* E *to* F *belongs to* $\mathscr{L}^{(d)}_{r,p}$, *then* T *belongs to* $\mathscr{A}^{(a)} - p$.

Proof. The proof is similar to the proof of Theorem 3.7. □

Remark 3.6. In particular, if we take the diagonal matrix $A = (a_{nk})$, where

$$a_{nk} = \begin{cases} n^{-\frac{1}{2}} : & k = n \\ 0 \quad : & k \neq n. \end{cases}$$

If a bounded linear operator T from E to F belongs to $S^{(d)}_p$, then T belongs to $\mathscr{A}^{(a)} - p$ for $0 < p < \infty$.

We now state the dual of the operator ideal formed by different s-number sequences.

Theorem 3.9. *The operator ideal* $\mathscr{A}^{(a)} - p$ *is symmetric and the operator ideal* $\mathscr{A}^{(h)} - p$ *is completely symmetric for* $0 < p \leq \infty$.

Proof. Since $a_n(T^{'}) \leq a_n(T)$ and $h_n(T^{'}) = h_n(T)$, for all $T \in \mathscr{L}(E,F)$, we have $\mathscr{A}^{(a)} - p \subseteq (\mathscr{A}^{(a)} - p)^{'}$ and $\mathscr{A}^{(h)} - p = (\mathscr{A}^{(h)} - p)^{'}$. □

Theorem 3.10. *Let* $0 < p \leq \infty$. *Then* $\mathscr{A}^{(c)} - p = (\mathscr{A}^{(d)} - p)^{'}$ *and* $\mathscr{A}^{(d)} - p \subseteq (\mathscr{A}^{(c)} - p)^{'}$. *In addition, if* T *belongs to the class of compact operators, then* $\mathscr{A}^{(d)} - p = (\mathscr{A}^{(c)} - p)^{'}$.

Proof. The proof follows from Theorem 2.2. □

Theorem 3.11. *Let* $0 < p \leq \infty$. *Then* $\mathscr{A}^{(x)} - p = (\mathscr{A}^{(y)} - p)^{'}$ *and* $\mathscr{A}^{(y)} - p = (\mathscr{A}^{(x)} - p)^{'}$.

Proof. The proof follows from Theorem 2.3. □

3.1 Small operator ideal

This section deals with the small ideals of operators. In [6], Pietsch proved that the ideal $S_p^{(a)}$ is small for $0 < p < \infty$. Here, we proved that the ideal formed by approximation type ces_p operators is small for $1 < p < \infty$.

Let A be a Cesàro matrix of order 1 and $\mathscr{A}^{(a)} - p$ be an ideal of approximation type ces_p operators.

Then we have the following theorem.

Theorem 3.12. *The quasi-Banach operator ideal* $\mathscr{A}^{(a)} - p$ *of approximation type* ces_p *operators is small for* $1 < p < \infty$.

Proof. Let $\lambda = \Big(\sum\limits_{n=1}^{\infty} \frac{1}{n^p}\Big)^{\frac{1}{p}}$ for $1 < p < \infty$. Then $[\mathscr{A}^{(a)} - p, \hat{\beta}^{(a)}_{A,p}]$ is a quasi-Banach operator ideal, where $\hat{\beta}^{(a)}_{A,p}(T) = \frac{1}{\lambda}\Big(\sum\limits_{n=1}^{\infty}\Big(\frac{1}{n}\sum\limits_{k=1}^{n} a_k(T)\Big)^p\Big)^{\frac{1}{p}}$. Let E, F be any two Banach spaces. Suppose that $\mathscr{A}^{(a)}_{(E \to F)} - p = \mathscr{L}(E,F)$, then there exists a constant $C > 0$ such that $\hat{\beta}^{(a)}_{A,p}(T) \leq C\|T\|$ for all $T \in \mathscr{L}(E,F)$. Assume that E and F both are infinite dimensional Banach spaces. Then by Dvoretzky's theorem [7] for $m = 1, 2, \cdots$ we have quotient spaces E/N_m and subspaces M_m of F which can be mapped onto l_2^m by isomorphisms X_m and A_m such that $\|X_m\|\|X_m^{-1}\| \leq 2$ and $\|A_m\|\|A_m^{-1}\| \leq 2$. Consider I_m be the identity map on l_2^m, Q_m be the quotient map from E onto E/N_m and J_m be the natural embedding map from M_m into F. Let a_n, d_n and u_n be approximation numbers, Kolmogorov numbers and Bernstein numbers [5], respectively. Then

$$1 = u_n(I_m) = u_n(A_m A_m^{-1} I_m X_m X_m^{-1})$$

$$
\begin{aligned}
&\le \|A_m\| u_n(A_m^{-1} I_m X_m) \|X_m^{-1}\| \\
&= \|A_m\| u_n(J_m A_m^{-1} I_m X_m) \|X_m^{-1}\| \\
&\le \|A_m\| d_n(J_m A_m^{-1} I_m X_m) \|X_m^{-1}\| \\
&= \|A_m\| d_n(J_m A_m^{-1} I_m X_m Q_m) \|X_m^{-1}\| \\
&\le \|A_m\| a_n(J_m A_m^{-1} I_m X_m Q_m) \|X_m^{-1}\| \qquad \text{for } n = 1, 2, \cdots, m.
\end{aligned}
$$

Now

$$
\begin{aligned}
&\sum_{k=1}^{n}(1) \le \sum_{k=1}^{n} \|A_m\| a_k(J_m A_m^{-1} I_m X_m Q_m) \|X_m^{-1}\| \\
\Rightarrow\ & \frac{1}{n}.n \le \|A_m\| \Big(\frac{1}{n}\sum_{k=1}^{n} a_k(J_m A_m^{-1} I_m X_m Q_m)\Big) \|X_m^{-1}\| \\
&\Rightarrow 1 \le \Big(\|A_m\|\|X_m^{-1}\|\Big)^p \Big(\frac{1}{n}\sum_{k=1}^{n} a_k(J_m A_m^{-1} I_m X_m Q_m)\Big)^p.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\Big(\sum_{n=1}^{m}(1)\Big)^{\frac{1}{p}} \le \Big(\|A_m\|\|X_m^{-1}\|\Big)\Big(\sum_{n=1}^{m}\Big(\frac{1}{n}\sum_{k=1}^{n} a_k(J_m A_m^{-1} I_m X_m Q_m)\Big)^p\Big)^{\frac{1}{p}} \\
\Rightarrow\ & \frac{1}{\lambda} m^{\frac{1}{p}} \le \|A_m\|\|X_m^{-1}\| \frac{1}{\lambda}\Big(\sum_{n=1}^{m}\Big(\frac{1}{n}\sum_{k=1}^{n} a_k(J_m A_m^{-1} I_m X_m Q_m)\Big)^p\Big)^{\frac{1}{p}} \\
\Rightarrow\ & \frac{1}{\lambda} m^{\frac{1}{p}} \le \|A_m\|\|X_m^{-1}\| \hat{\beta}^{(a)}_{A,p}(J_m A_m^{-1} I_m X_m Q_m) \\
&\qquad \le C\|A_m\|\|X_m^{-1}\|\|J_m A_m^{-1} I_m X_m Q_m\| \\
&\qquad \le C\|A_m\|\|X_m^{-1}\|\|J_m A_m^{-1}\|\|I_m\|\|X_m Q_m\| \\
&\qquad = C\|A_m\|\|X_m^{-1}\|\|A_m^{-1}\|\|X_m\| \\
&\qquad \le 4C.
\end{aligned}
$$

This is a contradiction as m is any arbitrary number. Thus E and F both cannot be infinite dimensional when $\mathscr{A}^{(a)}_{(E\to F)} - p = \mathscr{L}(E,F)$. This completes the proof. ☐

Theorem 3.13. *The quasi-Banach operator ideal $\mathscr{A}^{(d)} - p$ of Kolmogorov type ces_p operators is small for $1 < p < \infty$.*

Proof. The proof is similar to the proof of Theorem 3.12. ☐

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