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NONCONVEX MINIMIZATION IN GENERATING SPACE

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Abstract. A nonconvex minimization theorem has been established for generating of space of quasi 2-metric family with non commuting condition, which is defined as α-compatible.

1. Introduction and preliminary

The generating space of quasi 2-metric family is important because of its involvement with fuzzy and probabilistic 2-metric space and a minimization theorem [1], [2] is to obtain fixed point theorem. In this paper, we prove a non convex minimization theorem for generating space of quasi 2-metric family. Further, we prove fixed point theorem as an application of minimization theorem with non commuting condition Known as α-compatible. Generating space of quasi 2-metric family is defined as follows:-

Let X be a non empty set and $\{D_\alpha : \alpha \in (0,1]\}$ be family of mapping D_α from $XxXxX$ into R^+. $\{X, D_\alpha\}$ is called generating space of quasi 2-metric family if it satisfy following axioms:

(GM1) - For any two distinct points x and y there exists z in X such that $D_\alpha(x,y,z) \neq 0$, $\forall\ \alpha \in (0,1]$.

(GM2) - $D_\alpha(x,y,z) = 0$ if at least two x, y, z are equal and $\alpha \in (0,1]$.

(GM3) - $D_\alpha(x,y,z) = D_\alpha(x,z,y) = D_\alpha(z,y,x)$ for all x, y, z in X and $\alpha \in (0,1]$.

(GM4) - For any $\alpha \in (0,1]$ there exists $\alpha_1, \alpha_2, \alpha_3 \in (0,\alpha]$ such that $\alpha_1 + \alpha_2 + \alpha_3 \le \alpha$ and so $D_\alpha(x,y,z) \le D_{\alpha 1}(x,y,u) + D_{\alpha 2}(x,u,z) + D_{\alpha 3}(u,y,z)$.

(GM5) - $D_\alpha(x,y,z)$ is non increasing and left continuous in α and $\forall\ x$, y, z in X.

Throughout this paper, we assume that $K : (0,1] \to (0,\infty)$ is a non decreasing function satisfying the condition

$$K = \sup_\alpha K(\alpha).$$

Let E and F be mappings from generating space of quasi 2-metric family $\{X, D_\alpha\}$ into itself. The mapping E and F are said to be α-compatible if

$$\lim_{n\to\infty} D_\alpha(EFx_n, FEx_n, EEx_n) = 0 \text{ and } \lim_{n\to\infty} D_\alpha(EFx_n, FEx_n, FFx_n) = 0$$

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for all $\alpha \in (0,1]$ and x_n is a sequence in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some t in X.

Lemma 1. *Let E and F be α-compatible mappings from generating space of quasi 2-metric family $\{X, D_\alpha\}$ into itself. Suppose that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some t in X. Then*

(i) $\lim_{n\to\infty} EFx_n = Ft$ *if F is continuous.*

(ii) $EFt = FEt$ *and* $Et = Ft$ *if E and F are continuous.*

Proof. (i) Suppose that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some t in X. Since F is continuous, we have $\lim_{n\to\infty} FEx_n = Ft$. Now by (GM4), we have

$$D_\alpha(EFx_n, Ft, Ft) \le D_{\alpha 1}(EFx_n, Ft, FEx_n) + D_{\alpha 2}(EFx_n, FEx_n, Ft) + D_{\alpha 3}(FEx_n, Ft, Ft)$$

For $\alpha_1, \alpha_2, \alpha_3 \in (0,\alpha]$ such that $\alpha_1 + \alpha_2 + \alpha_3 \le \alpha$. On taking limit $n \to \infty$ we get $\lim_{n\to\infty} EFx_n = Ft$. Since E is continuous, $\lim_{n\to\infty} EFx_n = Et$ and also F is continuous so by (i) $\lim_{n\to\infty} EFx_n = Ft$. Hence by uniqueness $Et = Ft$. Now again $D_\alpha(EFt, FEt, EFt) \le \lim_{n\to\infty} D_\alpha(EFx_n, FEx_n, EFx_n)$ for all $\alpha \in (0,1]$. Since E and F and by (GM2) we get $EFt = FEt$.

2. Main result

Theorem 1. *Let $\{X, D'_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $F : X \to Y$ be a closed and $T : X \to X$ be continuous mapping satisfying*

(i) $D_\alpha(Tx, Ty, z) \le \max\{D_\alpha(Tx, y, z), D_\alpha(x, Ty, z), D_\alpha(x, y, Tz)\}$ *and*

(ii) $D'_\alpha(f(Tx), f(Ty), f(z)) \le \max\{D'_\alpha(f(Tx), f(y), f(z)), D'_\alpha(f(x), f(Ty), f(z)),$
$D'_\alpha(f(x), f(y), f(Tz))\}, \forall\ x, y, z$ *in X and* $\alpha \in (0,1]$

(iii) $\psi : \mathfrak{R} \to \mathfrak{R}$ *be a non decreasing continuous and bounded below function,*

(iv) $\phi : f(X) \to \mathfrak{R}$ *be a lower semi continuous and bounded below function,*

(v) *for any $p \in X$ with* $\inf_{x' \in X} \psi(\phi(f(x))) < \psi(\phi(f(p)))$ *there exists q with $p \ne Tq$ and*
$\max[\max\{D_\alpha(Tq, p, z), D_\alpha(q, Tp, z), D_\alpha(q, p, Tz)\}, c.\max\{D'_\alpha(f(Tq), f(p), f(z)),$
$D'_\alpha(f(q), f(Tp), f(z)), D'_\alpha(f(q), f(p), f(Tz))\}] \le K(\alpha)[\psi(\phi(f(p))) - \psi(\phi(f(q)))]$
$\forall\ x, y, z$ *in X and* $\alpha \in (0,1]$ *and c is any constant.*

Then there exists an x_0 in X such that $\inf_{x \in X} \psi(\phi(f(x))) = \psi(\phi(f(x_0)))$.

Proof. Let us suppose $\inf_{x\in X} \psi(\phi(f(x))) < \psi(\phi(f(y)))$ for every y in X and choose $r \in X$ for which $\psi(\phi(f(r)))$ is defined, then inductively we define a sequence $\{r_n\} \subset X$ with $r_1 = r$. Suppose r_n is known and consider

$$\begin{aligned} W_n = \Big\{\Big[w \in X : \max\Big[&\max\{D_\alpha(Tw, r_n, z), D_\alpha(w, Tr_n, z), D_\alpha(w, r_n, Tz)\}, \\ &c.\max\{D'_\alpha(f(Tw), f(r_n), f(z)), D'_\alpha(f(w), f(Tr_n), f(z)), D'_\alpha(f(w), f(r_n), f(Tz))\}\Big] \\ &\le K(\alpha)\Big[\psi(\phi(f(r_n))) - \psi(\phi(f(w)))\Big] \qquad \forall\ x, y, z \text{ in } X \text{ and } \alpha \in (0,1]. \end{aligned}$$

W_n is non empty set and there exists $w \in W_n$ such that $r_n \neq Tw$. We can choose $r_{n+1} \in W_n$ such that and $r_n \neq T(r_{n+1})$ and

$$\psi(\phi(f(r_{n+1}))) \leq \inf_{x\in X} \psi(\phi(f(x))) + 1/3\Big[\psi(\phi(f(r_n))) - \inf_{x\in X} \psi(\phi(f(x)))\Big].$$

Clearly $\psi(\phi(f(r_{n+1})))$ is a non increasing lower bounded sequence, hence it is a convergent sequence.

Now we prove $\{r_n\}$ and $\{f(r_n)\}$ are Cauchy sequences:

$$\begin{aligned}
&\max\{D_\alpha(Tr_n, Tr_{n+1}, w), c.D'_\alpha(f(Tr_n), f(r_{n+1}), f(w))\} \\
&\quad \leq \max\Big[\max\{D_\alpha(Tr_n, r_{n+1}, w), D_\alpha(r_n, Tr_{n+1}, w), D_\alpha(r_n, r_{n+1}, Tw)\}, \\
&\qquad c.\max\{D'_\alpha(f(Tr_n), f(r_{n+1}), f(w)), D'_\alpha(f(r_n), f(Tr_{n+1}), f(w)), \\
&\qquad\qquad D'_\alpha(f(r_n), f(r_{n+1}), f(Tw))\}\Big] \\
&\quad \leq K(\alpha)\Big[\psi(\phi(f(r_n))) - \psi(\phi(f(r_{n+1})))\Big]
\end{aligned}$$

$\forall\ n, m \in N,\ n < m \Rightarrow$ there exists $\alpha_j = \alpha_j(n,m)$; $\sum \alpha_j \leq \alpha$, such that

$$\begin{aligned}
&\max\Big\{\max\{D_\alpha(Tr_n, r_m, w), D_\alpha(r_n, Tr_m w), D_\alpha(r_n, r_m, Tw)\} \\
&\qquad c.\max\{D'_\alpha(f(Tr_n), f(r_m), f(w)), D'_\alpha(f(r_n), f(Tr_m), f(w)), \\
&\qquad\qquad D'_\alpha(f(r_n), f(r_m), f(Tw))\}\Big\} \\
&\quad \leq \sum_{j=n} \max\Big\{\max\{D_{\alpha j}(Tr_j, r_{j+1}, w), D_{\alpha j}(r_j, Tr_{j+1}, w), D_{\alpha j}(r_j, r_{j+1}, Tw)\} \\
&\qquad c.\max\{D'_{\alpha j}(f(Tr_j), f(r_{j+1}), f(w)), D'_{\alpha j}(f(r_j), f(Tr_{j+1}), f(w)), \\
&\qquad\qquad D'_{\alpha j}(f(r_j), f(r_{j+1}), f(Tw))\}\Big\}
\end{aligned}$$

Hence, $\forall\ n, m \in N,\ n < m$ and $\mu < \alpha$:

$$\begin{aligned}
&\max\Big\{\max\{D_\alpha(Tr_n, r_m, w), D_\alpha(r_n, Tr_m, w), D_\alpha(r_n, r_m, Tw)\} \\
&\qquad c.\max\{D'_\alpha(f(Tr_n), f(r_m), f(w)), D'_\alpha(f(r_n), f(Tr_m), f(w)), \\
&\qquad\qquad D'_\alpha(f(r_n), f(r_m), f(Tw))\}\Big\} \\
&\quad \leq K(\mu)\sum_{j=n}^{m-1}\Big[\psi(\phi(f(r_j))) - \psi(\phi(f(r_{j+1})))\Big] \\
&\quad \leq K(\alpha)\Big[\psi(\phi(f(r_n))) - \psi(\phi(f(r_m)))\Big]
\end{aligned}$$

for some α_j with $0 < \alpha_{j+1} < \alpha_k \leq \alpha,\ j = n, \ldots, \ldots, m-1$.

$$\begin{aligned}
D_\alpha(r_n, r_{n+1}, w) &\leq D_{\alpha 1}(r_n, r_{n+1}, Tr_{n+1}) + D_{\alpha 2}(r_n, Tr_{n+1}, w) + D_{\alpha 3}(Tr_{n+1}, r_{n+1}, w) \\
&\leq D_{\alpha 1}(r_n, r_{n+1}, Tr_{n+1}) + D_{\alpha 2}(r_n, Tr_{n+1}, w) + D_{\alpha 3}(Tr_{n+1}, r_{n+1}, Tr_n)
\end{aligned}$$

$$
\begin{aligned}
&+D_{\alpha 4}(Tr_{n+1}, Tr_n, w) + D_{\alpha 5}(Tr_n, r_{n+1}, w)\\
&\text{for } \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 \le \alpha\\
\le\ & 3\Big[\max\{D_\alpha(Tr_n, r_{n+1}, w), D_\alpha(r_n, Tr_{n+1}, w), D_\alpha(r_n, r_{n+1}, Tw)\},\\
&c.\max\{D'_\alpha(f(Tr_{rn}), f(r_{n+1}), f(w)), D'_\alpha(f(r_n), f(Tr_{n+1}), f(w)),\\
&\qquad D'_\alpha(f(r_n), f(r_{n+1}), f(Tw))\}\Big]\\
\le\ & 3K(\alpha)\Big[\psi(\phi(f(r_n))) - \psi(\phi(f(r_{n+1})))\Big].
\end{aligned}
$$

Then also we get

$$D_\alpha(r_n, r_m, w) \le 3K(\alpha)\Big[\psi(\phi(f(r_n))) - \psi(\phi(f(r_m)))\Big] \quad \text{where } n<m.$$

In the manner we obtain

$$D'_\alpha(f(r_n), f(r_{n+1}), f(w)) \le 3K(\alpha)\Big[\psi(\phi(f(r_n))) - \psi(\phi(f(r_m)))\Big] \quad \text{where } n<m.$$

Hence $\{r_n\}$ and $\{f(r_n)\}$ are Cauchy sequences.

Assume that $\lim_{n\to\infty} r_n = a$ and $\lim_{n\to\infty} f(r_n) = b$. Since f is closed therefore $f(a) = b$

By the continuity of Ψ and lower semi continuity of ϕ we have

$$\psi(\phi(b)) \le \lim_{n\to\infty}\psi(\phi(f(r_n))) = \lim_{n\to\infty}\psi(\phi(f(r_{n+1}))).$$

Let $\delta = \inf_{x\in X}\psi(\phi(f(x))) \in \Re$

$$\psi(\phi(f(r_{n+1}))) \le \inf_{x\in X}\psi(\phi(f(X))) + 1/3[\psi(\phi(f(r_n))) - \inf_{x\in X}\psi(\phi(f(x)))],$$

we have

$$\lim_{n\to\infty}\psi(\phi(f(r_{n+1}))) \le (2/3)\delta + 1/3\lim_{n\to\infty}\psi(\phi(f(r_n))) = (2/3)\delta + 1/3\lim_{n\to\infty}\psi(\phi(f(r_{n+1})))$$

$$\psi(\phi(f(a))) < \psi(\phi(b)) \le \lim_{n\to\infty}\psi(\phi(f(r_{n+1}))) \le \delta = \inf_{x\in X}\psi(\phi(f(x))) \le \psi(\phi(f(a)))$$

which is contradiction, therefore there exists x_0 in X such that

$$\inf_{x\in X}\psi(\phi(f(x))) = \psi(\phi(f(x_0))).$$

Now we give a fixed point theorem as an application of the above theorem under non commuting condition known as α-compatible.

Theorem 2. *Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $F : X \to Y$ be a closed and $T : X \to X$ be continuous mapping satisfying*

(i) $D_\alpha(Tx, Ty, z) \le \max\{D_\alpha(Tx, y, z), D_\alpha(x, Ty, z), D_\alpha(x, y, Tz)\}$ *and*

(ii) $D'_\alpha(f(Tx), f(Ty), f(z)) \le \max\{D'_\alpha(f(Tx), f(y), f(z)), D'_\alpha(f(x), f(Ty), f(z)), D'_\alpha(f(x), f(y), f(Tz))\}$,

(iii) $\psi : \Re \to \Re$ *be a non decreasing continuous and bounded below function,*

(iv) $\phi : f(X) \to \Re$ *be a lower semi continuous and bounded below function,*

(v) $S : X \to X$ *be a continuous mapping,*

(vi) *S and T are α compatible and*

$$\begin{aligned}\max\Big[&\max\{D_\alpha(Tx, TSx, z), D_\alpha(x, TSx, z), D_\alpha(x, Sx, Tz)\},\\ &c.\max\{D'_\alpha(f(Tx), f(TSx), f(z)), D'_\alpha(f(x), f(TSx), f(z)),\\ &D'_\alpha(f(x), f(Sx), f(Tz))\}\Big]\\ &\le K(\alpha)\Big[\psi(\phi(f(x))) - \psi(\phi(f(Sx)))\Big]\end{aligned}$$

$\forall$ x, y, z in X and $\alpha \in (0,1]$ and c is any constant. Then there exists unique common fixed point x_0 in X.

Proof. If $x_0 \in X$ such that $\inf_{x\in X} \psi(\phi(f(x))) = \psi(\phi(f(x_0)))$ then $x_0 = TSx_0$, $Sx_0 = Tx_0$ and also $x_0 = Tx_0$. By the hypothesis, if $x_0 \ne Tx_0$ or $Sx_0 \ne Tx_0$. Then for some $\alpha \in (0,1]$

$$\begin{aligned}0 &< \max\Big\{D_\alpha(Tx_0, TSx_0, z), D_\alpha(x_0, TSx_0, z), D_\alpha(x_0, Sx_0, Tz)\Big\}\\ &\le K(\alpha)\Big[\psi(\phi(f(x_0))) - \psi(\phi(f(Sx_0)))\Big] \le 0\end{aligned}$$

which contradiction. Then $Sx_0 = Tx_0$. Now by α compatibility of S and T and by the lemma we get $Sx_0 = TSx_0 = STx_0 = Tx_0$.
Also for $\alpha_1, \alpha_2, \alpha_3 \in (0,\alpha]$ such that $\alpha_1 + \alpha_2 + \alpha_3 \le \alpha$

$$\begin{aligned}D_\alpha(x_0, Tx_0, z) &\le D_{\alpha 1}(x_0, Tx_0, TSx_0) + D_{\alpha 2}(x_0, TSx_0, z) + D_{\alpha 3}(TSx_0, Tx_0, z)\\ &\le D_{\alpha 3}(TSx_0, Tx_0, z) = 0. \quad \text{Hence } \ Tx_0 = Sx_0 = x_0.\end{aligned}$$

Uniqueness. Let us assume there exists another fixed point y_0 such that $Sy_0 = Ty_0 = y_0$ and by the Theorem 1 we have $\inf_{x\in X} \psi(\phi(f(x))) = \psi(\phi(f(y_0)))$.
But $\inf_{x\in X} \psi(\phi(f(x))) = \psi(\phi(f(x_0)))$ hence by uniqueness of infima we get $x_0 = y_0$.

Corollary 1. *Let $\{X, D_\alpha : \alpha \in (0,1]\}$ and $\{Y, D'_\alpha : \alpha \in (0,1]\}$ be two complete generating space of quasi 2-metric family. $F : X \to Y$ be a closed, $\phi : f(X) \to \Re$ be a lower semi continuous and bounded below function. Let $S : X \to X$ be a mapping such that*

$$\max\Big\{D_\alpha(Sx, x, z), c.D'_\alpha(f(Sx), f(x), f(z))\Big\} \le K(\alpha)\Big[\phi(x) - \phi(Sx)\Big]$$

$\forall$ x, y, z in X and $\alpha \in (0,1]$ and c is any constant. Then there exists $Sx_0 = x_0$.

Proof. Considering $T = I$ and $\psi = I$ we get required result.

3. Example

Let $X = [0,1]$ and $Y = [0,\infty]$, $D_\alpha = D'_\alpha = D_1$ defined by $D_1(x,y,z) = \dfrac{D(x,y,z)}{1+D(x,y,z)}$ and $D(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}$. The mappings are defined as follows: $T : X \to X$ as $Tx = x^2$, $f : X \to X$ as $fx = x$, $\phi : f(X) \to R$ as $\phi(x) = 1/(1-x)$ and $\psi : R \to R$, $\psi(x) = x^2/2$ and $K(\alpha) = 3$ satisfy the all conditions of Theorem 1. Also $S : X \to X$ is defined $Sx = x^3/2$, then (S,T) is α compatible which satisfying the condition of Theorem 2 and hence 0 is a unique fixed point.

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References

[1] D. Downing and W. A. Kirk, *A generalization of Cariti's theorem with applications in nonlinear mapping theory*, Pacific J. Math. **69**(1977), 339–346.

[2] J. S. Jung, Y. J. Cho and J. K. Kim, *Minimization theorems for fixed point theorems in fuzzy metric space and application*, Fuzzy Sets and System **61**(1994), 199–270.

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