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ON GRADED SECOND MODULES

H. ANSARI-TOROGHY AND F. FARSHADIFAR

Abstract. This paper deals with some results concerning graded second modules.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity.

A proper submodule N of an R-module M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in N$, we have $m \in N$ or $r \in (N :_R M)$ [6].

In [7], I.G. Macdonald introduced the notion of secondary modules. A non-zero R-module M is said to be *secondary* if for each $a \in R$ the endomorphism of M given by multiplication by a is either surjective or nilpotent [7].

In [11], S. Yassemi introduced the dual notion of prime submodules (i.e., second submodules) and investigated some properties of this class of modules. A non-zero submodule N of an R-module M is said to be *second* if for each $a \in R$ the homomorphism $N \xrightarrow{a} N$ is either surjective or zero. This implies that $Ann_R(N) = P$ is a prime ideal of R and S is said to be P-*second* [11]. More information about of this class of modules can be found in [2] and [3].

Let G be a group with identity e. The ring R graded by the group G will be denoted by $R = \oplus_{g\in G} R_g$, where R_g is an additive subgroup of R and $R_g.R_h \subseteq R_{gh}$ for every g, h in G. If an element of R belongs to $\cup_{g\in G} R_g = h(R)$, then it is called *homogeneous* and any $x_g \in R_g$ is said to *have degree* g. In the rest of this paper let R be a G-graded ring. An R-module M is said to be a *graded module* if $M = \oplus_{g\in G} M_g$ for a family of subgroups $\{M_g\}_{g\in G}$ of M such that $R_g.M_h \subseteq M_{gh}$ for every g, h in G. A *graded submodule* N of M is a submodule verifying $N = \oplus_{g\in G}(N \cap M_g)$. Moreover, M/N becomes a graded R-module with $(M/N)_g = (M_g + N)/N$. In this case, M/N is called a *gr-quotient* of M. Also if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called *homogeneous*. Let $M = \oplus_{g\in G} M_g$ and $N = \oplus_{g\in G} N_g$ be graded R-modules. An R-homomorphism $f : M \to N$ is said to be a gr-homomorphism of degree h,

Corresponding author: H. Ansari-Toroghy.
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$h \in G$, if $f(M_g) \subseteq N_{gh}$ for all $g \in G$. Graded homomorphisms of degree h build an additive subgroup $HOM_R(M,N)_h$ of $Hom_R(M,N)$. It is clear that $HOM_R(M,N) = \oplus_{h\in G} HOM_R(M,N)_h$ is a graded abelian group of type G. The category of graded R-modules has graded R-modules as objects. A morphism in this category is an R-module homomorphism of degree e. Given a multiplicatively closed subset $S \subseteq h(R)$, the ring of fraction $S^{-1}R$ turns into a ring graded by G means of

$$(S^{-1}R)_g = \{r/s : s \in S, r \in h(R) \quad \text{and} \quad g = degr - degs\}$$

for every $g \in G$. Recall that $S^{-1}M$ can be defined as $S^{-1}R \otimes_R M$, when M is an R-module.

A graded ideal P of R is said to be *graded prime*, more briefly, *gr-prime* if $P \neq R$ and whenever $ab \in P$, we have $a \in P$ or $b \in P$, where $a, b \in h(R)$. The *graded radical* of a graded ideal I of R, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A graded submodule N of a graded R-module M is said to be *gr-prime* (resp. *gr-primary*) if $N \neq M$ and whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a \in (N :_R M)$ (resp. $a \in Gr((N :_R M))$) [4]. This implies that $Ann_R(M/N) = P$ (resp. $Gr(Ann_R(M/N)) = P$) is a gr-prime ideal of R and N is said to be P-*gr-prime* (resp. P-*gr-primary*). Also, a graded R-module M is said to be *gr-prime* if the zero submodule of M is a *gr-prime* submodule of M.

Let M be a non-zero graded R-module. Then M is said to be a *gr-second* (resp. *gr-secondary*) if for each homogeneous element a of R, the endomorphism of M given by multiplication by a is either surjective or zero (resp. nilpotent) [1] (resp. [10]). This implies that $Ann_R(M) = P$ (resp. $Gr(Ann_R(M)) = P$) is a gr-prime ideal of R and M is said to be P-*gr-second* (resp. P-*gr-secondary*). For convenience, a graded submodule of M which is gr-second (resp. gr. secondary), is called a *gr-second* (resp. *gr-secondary*) *submodule* of M.

The purpose of this paper is to obtain some results concerning graded second submodules. Most of results are related to reference [11] which have been proved for second submodules.

2. Main results

Remark 2.1. It is clear that every second R-module which is a graded module is a gr-second R-module but the converse is not true in general. For example, if we take $G = \mathbb{Z}$ and $R = K[x,x^{-1}](= K[x]_x)$, where K is a field and x is an indeterminate, graded in the obvious way, R as an R-module is graded simple [5, 1.5.14(c)]. Hence R is a gr-second R-module. But R is not a secondary R-module by [10]. Hence R is not a second R-module.

Lemma 2.2 (See [9]). (a) *If I is a graded ideal of R, then $I \subseteq Gr(I)$.*

(b) *If I and J are graded ideals of R such that $I \subseteq J$, then $Gr(I) \subseteq Gr(J)$.*

(c) *If P is a gr-prime ideal of R, then $Gr(P^n) = P$ for all $n > 0$.*

A graded submodule N of a graded R-module M is said to be *gr-minimal* if it is minimal in the lattice of graded submodules of M [8].

Proposition 2.3. *Let M be a graded R-module. Then the following hold.*

(a) *If S is a gr-secondary submodule of M, then S is gr-second if and only if $Ann_R(S)$ is a gr-prime ideal of R.*

(b) *Let S be a graded submodule of a P-gr-second module M. Then S is a P-gr-secondary submodule if and only if S is a P-gr-second submodule.*

(c) *If S is a gr-minimal submodule of M, then S is a gr-second submodule of M.*

Proof. (a) This is obvious.

(b) Assume S is a P-gr-secondary submodule of M. Then $P = Ann_R(M) \subseteq Ann_R(S) \subseteq Gr(Ann_R(S)) = P$ by using Lemma 2.2 (a). Thus $P = Ann_R(S)$. Now the assertion follows from part (a). The reverse implication is clear.

(c) Let S be a gr-minimal submodule of M. Since for each $r \in h(R)$, rS is a graded submodule of M, by assumption, $rS = 0$ or $rS = S$ as desired. □

Proposition 2.4. *Let P be a gr-prime ideal of R. Then the following hold.*

(a) *The sum of P-gr-second R-modules is a P-gr-second R-module.*

(b) *Every product of P-gr-second R-modules is a P-gr-second R-module.*

(c) *Every non-zero gr-quotient of a P-gr-second R-module is a P-gr-second R-module.*

Proof. We only prove the part (a). The proofs of parts (b) and (c) are similar.

(a) Let $M_1, M_2, \ldots, M_n$ be P-gr-second R-modules. Then for each $1 \le i \le n$ we have $Ann_R(M_i) = P$ and hence $Ann_R(\sum_{i=1}^n M_i) = P$. If $r \in h(R) - P$, then $rM_i = M_i$. Hence $r(\sum_{i=1}^n M_i) = \sum_{i=1}^n M_i$, as desired. □

Lemma 2.5. *Let P be a graded prime ideal of R and let S be a non-zero graded submodule of a graded R-module M. Then the following are equivalent.*

(a) *S is a P-gr-second submodule of M.*

(b) *$W^{gr}(S) \subseteq Ann_R(S) = P$, where*

$$W^{gr}(S) = \{a \in h(R) : \text{ the homothety } S \xrightarrow{a} S \text{ is not surjective}\}.$$

Proof. It is straightforward. □

A graded R-module M is said to be *gr-divisible* if $ax = m$ with $a \in h(R)$ and $m \in h(M)$, has a solution in M [8].

Theorem 2.6. *Let M be a graded R-module and let S be a non-zero graded submodule of M satisfying that $Ann_R(S) = P$ is a graded prime ideal of R. Then the following are equivalent.*

(a) *S is a P-gr-second submodule of M.*

(b) *S is a gr-divisible R/P-module.*

(c) *$rS = S$ for all $r \in h(R) - P$.*

(d) *$IS = S$ for all graded ideals I with $I \nsubseteq P$.*

(e) *$W^{gr}(S) \subseteq P$.*

Proof. $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (e)$ are straightforward.

$(e) \Rightarrow (a)$. By Lemma 2.5. □

Definition 2.1. Let P be a graded prime ideal of R. A graded submodule N of a graded R-module M is called a *minimal P-gr-secondary* (resp. *P-gr-second*) submodule of M if N is a P-gr-secondary (resp. P-gr-second) submodule which contains no other P-gr-secondary (resp. P-gr-second) submodules of M.

Theorem 2.7. *Let M be a graded R-module. Then a submodule N of M is minimal P-gr-secondary if and only if N is a minimal P-gr-second submodule of M.*

Proof. $(\Leftarrow)$. By Proposition 2.3 (b).

$(\Rightarrow)$. Assume that N is a minimal P-gr-secondary submodule of M. If $r \in W^{gr}(N)$, then $rN \neq N$. Since rN is a graded quotient of N, we have that rN is a P-gr-secondary submodule of N. As N is a minimal P-gr-secondary submodule of M, $rN = 0$ so that $r \in Ann_R(N)$. Therefore, $W^{gr}(N) \subseteq Ann_R(N)$. Thus N is a P-gr-second submodule of M by using Lemma 2.5. Now the result follows from Proposition 2.3 (b). □

R is said to be a *gr-field* if every nonzero homogeneous element of R is invertible.

A graded R-module M is said to be *gr-injective* if it is an injective object in the category of graded R-modules.

A graded R-module M is said to be *graded torsion-free* if $a \in h(R)$ and $m \in M$ with $am = 0$ implies that either $m = 0$ or $a = 0$ [4].

Theorem 2.8. *Let M be a gr-prime module. Then the following are equivalent.*

(a) *M is a gr-second module.*

(b) *M is a gr-injective $R/Ann_R(M)$-module.*

Proof. Since M is a gr-prime module, we have that $P = Ann_R(M)$ is a gr-prime ideal of R by [4, 2.7] and M is a gr-torsion-free R/P-module by [4, 2.11]. Hence the graded R/P-homomorphism $\phi : M \to S^{-1}M$ given by $\phi(m) = m/1$, where $S = h(R/P) - 0$, is a monomorphism.

$(a) \Rightarrow (b)$. Since M is a P-gr-second module, we have that M is a gr-divisible R/P-module by Theorem 2.6. This implies that ϕ is an isomorphism. Hence M is an $S^{-1}(R/P)$-module. As $S^{-1}(R/P)$ is a gr-field and M is a gr-divisible $S^{-1}(R/P)$-module by [8, B.II.2], it is easy to see by a similar argument as the ungraded case that M is a gr-injective R/P-module.

$(b) \Rightarrow (a)$. Since M is a gr-injective R/P-module, we have that M is a gr-divisible R/P-module. Thus we have that M is gr-second by Theorem 2.6. □

Proposition 2.9. *Let M be a graded R-module and let N be a graded submodule of M. Then we have the following.*

(a) *If M is a gr-primary module and N is a gr-second submodule of M, then N is $Ann_R(N)$-gr-primary.*

(b) *If M is a gr-prime module and N is a gr-second submodule of M, then $rN = rM \cap N$ for each $r \in h(R)$.*

(c) *If $Ann_R(N)$ is a gr-prime ideal of R and N is a gr-minimal in the set of all graded submodules K of M such that $Ann_R(K) = Ann_R(N)$, then N is a gr-second submodule of M.*

Proof. (a) First we note that as N is a gr-second submodule of M, $Gr(Ann_R(N)) = Ann_R(N)$ by Lemma 2.2 (c). Now let $rm \in N$, where $r \in h(R) - Ann_R(N)$ and $m \in h(M)$. Since N is a gr-second submodule of M, we have $rN = N$. Thus $rm = rn$ for some $n \in N$. As $r \notin Gr(Ann_R(N))$, we have $r \notin Gr(Ann_R(M))$ by Lemma 2.2 (b). As M is gr-primary, we have that $m \in N$ as required.

(b) Let $r \in h(R)$ and let $rm \in N$. Since N is gr-second, $rN = 0$ or $rN = N$. If $rN = 0$, we have $r \in Ann_R(M)$ because M is gr-prime. Hence $rN = rM \cap N = 0$. If $rN = N$, then $rm = rn$ for some $n \in N$. Since M is gr-prime and $r \notin Ann_R(N)$, we have $m = n$. Thus $rm \in rN$. Therefore $rM \cap N = N \subseteq rN$. Thus $rM \cap N = N = rN$ because the reverse inclusion is clear.

(c) As $Ann_R(N)$ is gr-prime, $N \neq 0$. Let $r \in h(R)$ and $rN \neq N$. Since rN is a graded submodule of M, the claim is obviously true in the case that $Ann_R(rN) = Ann_R(N)$ by assumption. So we assume that $Ann_R(rN) \not\subseteq Ann_R(N)$. Then there exists $s \in h(Ann_R(rN))$ such that $s \notin Ann_R(N)$. Hence $srN = 0$. Since $Ann_R(N)$ is gr-prime, it follows that $rN = 0$, as desired. □

A graded R-module M is said to be *graded injective cogenerator* if it is injective cogenerator object in the category of graded R-modules.

Theorem 2.10. *Let E be a graded injective cogenerator of R and let N be a graded submodule of a graded R-module M. Then N is a gr-prime submodule of M if and only if $HOM_R(M/N,E)$ is a gr-second R-module.*

Proof. Let N be a gr-prime submodule of M and let $r \in h(R)$. Then $M/N \neq 0$ if and only if $HOM_R(M/N,E) \neq 0$ by using similar arguments as the ungraded case. Further, $M/N \xrightarrow{r} M/N$ is either injective or zero if and only if

$$HOM_R(M/N,E) \xrightarrow{r} HOM_R(M/N,E)$$

is either surjective or zero by using similar arguments as the ungraded case. □

A graded submodule N of a graded R-module M is said to be *gr-maximal* if it is maximal in the lattice of graded submodules of M [8].

Theorem 2.11. *Let R be an integral domain which is not a gr-field and K the gr-field of quotients of R. Then the R-module K has no gr-minimal submodule and K is the only gr-second submodule of K.*

Proof. Since $(0:_K r) = 0$ for every non-zero element $r \in h(R)$, we have $Ann_R(N) = 0$ for every non-zero graded submodule N of M. Consequently, K has no gr-minimal submodule, for if L is a gr-minimal submodule of K, then $Ann_R(L)$ is a gr-maximal ideal of R. But since R is not a gr-field, $Ann_R(L) \neq 0$, which is a contradiction. Clearly K is a 0-gr-second submodule of K. To show that K is the only gr-second submodule of K, we assume the contrary and let S be a proper gr-second submodule of K. Since S is proper, there exists $y/u \in h(K)$ and $y/u \notin S$. This implies that $1/u \notin S$. There exists $0 \neq x/t \in h(S)$ because S is gr-second. Since $Ann_R(S) = 0$, we have $uS = S$. Thus $x/t = u(z/h)$ for some $z/h \in S$. It follows that $x/u = (tz)/h \in S$. Now $xS = S$ implies that $x/u = xw$ for some $w \in S$. Since $x \neq 0$, it follows that $1/u = w \in S$, which is a contradiction. □

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References

[1] H. Ansari-Toroghy and F. Farshadifar, *Graded comultiplication modules*, Chiang Mai J. Sci., to appear.

[2] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime submodules*, Algebra Colloq., to appear.

[3] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of prime submodules (II)*, Mediterr. J. Math., to appear.

[4] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci. **33** (2006), 3–7.

[5] W. Bruns and J. Herzong, *Cohen-Macaulay Rings*, **39**, Cambridge studies in Advanced Mathematics, 1996.

[6] J. Dauns, *Prime submodules*, J. Reine Angew. Math., **298** (1978), 156–181.

[7] I.G. Macdonald, *Secondary representation of modules over a commutative ring*, Sympos. Math., **XI** (1973), 23–43.

[8] C. Nastasescu and F. Van Oystaeyen, Graded Ring Theory, Mathematical Library 28, North Holand, Amsterdam, 1982.

[9] M.A. Refai, M. Hailat, and S. Obiedat, *Graded radicals and graded prime spectra*, Far East J. Math. Sci. (FJMS), Part I (2000), 59-73.

[10] R. Y. Sharp, *A symptotic behavior of certain sets of attached prime ideals*, J. London Math. Soc., **34** (1986), 212–218.

[11] S. Yassemi, *The dual notion of prime submodules*, Arch. Math (Brno) **37** (2001), 273–278.

Department of Mathematics, Faculty of Science, Guilan University, P. O. Box 1914, Rasht, Iran.

E-mail: ansari@guilan.ac.ir

Department of Mathematics, Faculty of Science, Guilan University, P. O. Box 1914, Rasht, Iran.

E-mail: farshadifar@guilan.ac.ir