SOME RESULTS ON COMMON FIXED POINTS AND BEST APPROXIMATION

ABDUL RAHIM KHAN, ABDUL LATIF, ARJAMAND BANO
AND NAWAB HUSSAIN

Abstract. We prove a common fixed point result for \((f, g)\)-nonexpansive maps and then derive certain results on best approximation. Our results generalize the results of Al-Thagafi [1], Jungck and Sessa [6], Khan, Hussain and Thaheem [8], Latif [9, 10] and Sahab, Khan and Sessa [12].

1. Preliminaries.

Let \(X\) be a linear space. A \(p\)-norm on \(X\) is a real valued function \(\|\cdot\|_p\) on \(X\) with \(0 < p \leq 1\), satisfying the following conditions:

(i) \(\|x\|_p \geq 0\) and \(\|x\|_p = 0 \iff x = 0\)

(ii) \(\|\lambda x\|_p = |\lambda| \|x\|_p\)

(iii) \(\|x + y\|_p \leq \|x\|_p + \|y\|_p\)

for all \(x, y \in X\) and all scalars \(\lambda\). The pair \((X, \|\cdot\|_p)\) is called a \(p\)-normed space. It is a metric space with a translation-invariant metric \(d_p\) defined by \(d_p(x, y) = \|x - y\|_p\) for all \(x, y \in X\). If \(p = 1\), we obtain the concept of a normed space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some \(p\)-norm, \(0 < p \leq 1\) [9, 10]. The space \(l_p\) and \(L^p\), \(0 < p \leq 1\) are \(p\)-normed spaces. A \(p\)-normed space is not necessarily a locally convex space.

Let \(C\) be a subset of \(p\)-normed space \(X\), and let \(f, g\) be self maps of \(X\). The set \(C\) is called starshaped with respect to \(q \in C\) if for all \(x \in C\) and for all \(k, 0 \leq k \leq 1, kx + (1-k)q \in C\). The set \(C\) is said to be starshaped if it is starshaped with respect to one of its elements.

Each convex set is necessarily starshaped. For any \(x_0 \in X\), define \(d_p(x_0, C) = \inf_{u \in C} \|x_0 - u\|_p\), \(P_C(x_0) = \{y \in C : \|y - x_0\|_p = d_p(x_0, C)\}\) the set of best \(C\)-approximants to \(x_0\) and \(D_f^C(x_0) = \{y \in C : f(y) \in P_C(x_0)\}\). Note that \(P_C(x_0)\) contains \(f(D_f^C(x_0))\) and \(g(D_g^C(x_0))\). Assume that \(D_f^{C,g}(x_0) = D_f^C(x_0) \cap D_g^C(x_0)\) and \(D = P_C(x_0) \cap D_f^{C,g}(x_0)\).

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Let $\mathcal{S} = \{h_\alpha\}_{\alpha \in C}$ be a family of functions from $[0, 1]$ into $C$ such that $h_\alpha(1) = \alpha$ for each $\alpha \in C$. The family $\mathcal{S}$ is said to be contractive if there exists a function $\varphi : (0, 1) \to (0, 1)$ such that for all $\alpha, \beta \in C$ and all $t \in (0, 1)$ we have
\[
\|h_\alpha(t) - h_\beta(t)\|_p \leq |\varphi(t)|^p \|\alpha - \beta\|_p
\]
The family $\mathcal{S}$ is said to be jointly continuous (resp. jointly weakly continuous) if $t \to t_0$ in $[0, 1]$ and $\alpha \to \alpha_0$ in $C$ (resp. $t \to t_0$ in $[0, 1]$ and $\alpha \to \alpha_0$ in $C$), then $h_\alpha(t) \to h_{\alpha_0}(t_0)$ (resp. $h_\alpha(t) \rightharpoonup h_{\alpha_0}(t_0)$) in $C$; here $\to$ and $\rightharpoonup$ denote the strong and weak convergence respectively.

It is known that if $C \subseteq X$ is $q$-starshaped and $h_\alpha(t) = (1-t)q + tx$, $(x \in C; t \in [0, 1])$, then $\mathcal{S} = \{h_\alpha\}_{\alpha \in C}$ is a contractive jointly continuous family with $\varphi(t) = t$. Thus the class of subsets of $X$ with the property of contractiveness and jointly continuity contains the class of starshaped sets which in turns contains set class of convex sets (see, [3] and [8]). If for a subset $C$ of $X$, there exists a contractive jointly continuous family of functions $\mathcal{S} = \{h_\alpha\}_{\alpha \in C}$, then we say that $C$ has the property of contractiveness and joint continuity.

A map $T : X \to X$ is $(f, g)$-contraction [7, 9], if there exists a real number $k \in (0, 1)$ such that
\[
\|Tx - Ty\|_p \leq k\|fx - gy\|_p \quad \text{for all } x, y \in X.
\]
If in the above inequality $k = 1$, then $T$ is called $(f, g)$-nonexpansive. Also if $f = g$, we say that $T$ is $f$-nonexpansive. We denote the boundary of $C$ by $\partial C$, closure of $C$ by $cl(C)$ and the set of fixed points of $T$ by $F(T)$.

Using fixed point theory, Brosowski [2], Meinardus [11] have established some interesting results on invariant approximation in the setting of normed spaces. Jungck and Sessa [6] have also obtained some results in approximation theory in the setting of normed spaces. Their work has been extended, generalized and unified by many authors; for example, see [4, 5, 6, 8, 12, 13].

Recently, Latif [9] has obtained the following results on common fixed points and best approximation, which generalize and extend the recent work of Al-Thagafi [1], Sahab, Khan and Sessa [12] and Kaneko [7] etc.

**Theorem 1.1.** Let $X$ be a $p$-normed space and $C$ a closed subset of $X$ which is starshaped with respect to $q$. Let $f$, $g$ and $T$ be continuous self-maps of $C$ such that $T(C) \subseteq f(C) \cap g(C)$, $T$ commutes with $f$ and $g$, and $q \in F(f) \cap F(g)$. If $cl(T(C))$ is compact, $f$ and $g$ are affine, and $T$ is $(f, g)$-nonexpansive, then $F(f) \cap F(g) \cap F(T) \neq \emptyset$.

**Theorem 1.2.** Let $f$, $g$ and $T$ be self-maps of a $p$-normed space $X$ such that $x_0 \in F(f) \cap F(g) \cap F(T)$ and $C \subseteq X$ with $T(\partial C \cap C) \subseteq C$. Let $T$ be a $(f, g)$-nonexpansive map and continuous on $D \cup \{x_0\}$ such that $cl(T(D))$ is compact, and let $f$ and $g$ be continuous, surjective, affine and commute with $T$ on $D$. If $D$ is closed and starshaped with respect to $q \in F(f) \cap F(g)$, then $P_C(x_0) \cap F(f) \cap F(g) \cap F(T) \neq \emptyset$.

We also recall the following common fixed point result due to Jungck and Sessa [6].
Theorem 1.3. Let $f$ be a continuous self-map of a compact metric space $(X, d)$. If $fx \neq fy$ implies $d(fx, fy) < d(gx, hy)$ for some self-maps $g$ and $h$ of $X$, commuting with $f$, then $F(f) \cap F(g) \cap F(h)$ is singleton.

The aim of this paper is to extend Theorem 1.1 and 1.2 to a domain which is not necessarily starshaped. An approximation result for two maps when underlying best approximation set $D$ need not be compact, will be established. Our results extend the corresponding results of Jungck and Sessa [6], Khan, Hussain and Thaheem [8], Latif [9, 10], Sahab, Khan and Sessa [12], Habiniak [4] and Singh [13].

2. Common Fixed Points and Best Approximations

Our first result generalizes Theorem 1.1, in the sense that the underlying domain $C$ need not be starshaped.

Theorem 2.1. Let $f$, $g$ and $T$ be continuous self-maps on a subset $C$ of a $p$-normed space $X$. Suppose that $T$ is $(f, g)$-nonexpansive and commutes with $f$ and $g$ on $C$. Suppose that $C$ is compact and has a contractive jointly continuous family $\exists = \{h_x\}_{x \in C}$ such that $f(h_x(\alpha)) = h_{f(x)}(\alpha)$ and $g(h_x(\alpha)) = h_{g(x)}(\alpha)$ for all $x \in C$ and $\alpha \in [0, 1]$. Then $F(f) \cap F(g) \cap F(T) \neq \emptyset$.

Proof. For $n \in N$, let $\lambda_n = n(n + 1)^{-1}$. Define

$$T_n(x) = h_{T(x)}(\lambda_n), \quad x \in C.$$ 

Then each $T_n$ is a well-defined self-map of $C$. Since $f$ and $T$ commute on $C$, it follows from the property of $\exists$ that

$$T_n(fx) = h_{fT(x)}(\lambda_n) = h_{fT(z)}(\lambda_n) = f(h_{T(z)}(\lambda_n)) = fT_n(x) \quad (x \in C).$$

Thus for each $n$, $T_n$ commutes with $f$. Similarly, for each $n$, $T_n$ commutes with $g$.

Moreover, since $T$ is $(f, g)$-nonexpansive, we get

$$\|T_n x - T_n y\|_p = \|h_{T(z)}(\lambda_n) - h_{T(y)}(\lambda_n)\|_p$$

$$\leq [\varphi(\lambda_n)]^p \|T(x) - T(y)\|_p$$

$$\leq [\varphi(\lambda_n)]^p \|fx - gy\|_p$$

Then it follows from Theorem 1.3 that there exists $x_n \in C$ such that $x_n \in F(T_n) \cap F(f) \cap F(g)$ for each $n$. Thus, by the compactness of $C$ it follows that the sequence, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to some $z \in C$ and hence $T(x_{n_i}) \to T(z)$. The joint continuity of $\exists$ gives

$$x_{n_i} = T_{n_i}(x_{n_i}) = h_{T(x_{n_i})}(\lambda_{n_i}) \to h_{T(z)}(1) = Tz$$
and thus the uniqueness of the limit implies $Tz = z$. Also, since $fx_{n_i} = x_{n_i} = gx_{n_i} \to z$
using the continuity of $f$ and $g$ and the uniqueness of the limit, we have $f(z) = g(z) = z$, which completes the proof.

**Theorem 2.2.** Let $f$, $g$ and $T$ be self-maps of a $p$-normed space $X$ such that $x_0 \in F(f) \cap F(g) \cap F(T)$ and $C \subset X$ with $T(\partial C \cap C) \subset C$. Let $T$ be $(f, g)$-nonexpansive on $D \cup \{x_0\}$ and let $f$ and $g$ be continuous, surjective and commute with $T$ on $D$. Suppose that $D$ is compact and has a contractive jointly continuous family $\mathcal{F} = \{h_x\}_{x \in D}$ such that $f(h_x(\alpha)) = h_f(x)(\alpha)$ for all $x \in D$ and $\alpha \in [0, 1]$. Then $P_C(x_0) \cap F(f) \cap F(g) \cap F(T) \neq \varnothing$.

**Proof.** Let $y \in D$. Then $\|y - x_0\|_p = d_p(x_0, C)$ since $y \in P_C(x_0)$. Note that for any $k \in (0, 1)$
$$
\|kx_0 + (1 - k)y + x_0\|_p = (1 - k)\|y - x_0\|_p < d_p(x_0, C),
$$
It follows that the line segment $\{kx_0 + (1 - k)y : 0 < k < 1\}$ and the set $C$ are disjoint. Thus $y$ is not in the interior of $C$ and so $y \in \partial C \cap C$. Since $T(\partial C \cap C) \subset C$, $Ty$ must
be in $C$. Also since $fy \in P_C(x_0)$, $x_0 \in F(T) \cap F(g)$ and $T$ is $(f, g)$-nonexpansive on $D \cup \{x_0\}$, we have
$$
\|Ty - x_0\|_p = \|Ty - Tx_0\|_p \leq \|fy - gx_0\|_p = \|fy - x_0\|_p = d_p(x_0, C),
$$
and hence $Ty \in P_C(x_0)$. Moreover, since $T$ commutes with $f$ on $D$ and $x_0 \in F(g) \cap F(T)$, we have
$$
\|fTy - x_0\|_p = \|fTy - Tx_0\|_p \leq \|f^2y - gx_0\|_p = \|f^2y - x_0\|_p = d_p(x_0, C),
$$
and similarly, since $T$ commutes with $g$ and $x_0 \in F(f) \cap F(T)$, we get
$$
\|gTy - x_0\|_p \leq \|g^2y - x_0\|_p = d_p(x_0, C).
$$
Thus $fTy$ and $gTy$ are in $P_C(x_0)$ and so $Ty \in D_T^f(x_0)$. Thus the definition of $D$ implies that $Ty \in D$. Consequently $T$ maps $D$ into $D$. For $n \in N$, define the maps $T_n$ from $D$ into $D$ as in the proof of Theorem 2.1. As in the proof of Theorem 2.1, there exists $x_n \in D$ such that $x_n \in F(T_n) \cap F(f) \cap F(g)$ for each $n$. In particular $x_n = f_x = gx_n$, by compactness of $D$, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to $z \in D$ and hence $T(x_{n_i}) \to Tz$. Now, we complete the proof as that of Theorem 2.1.

Now if we take $f = g$ in Theorem 2.2, then $D = P_C(x_0) \cap D_T^f(x_0)$ and thus we obtain the following result which generalizes Theorem 3.2 of Al-Thagafi [1] in the sense that the best approximation set $D$ in it need not be starshaped.

**Corollary 2.3.** Let $f$, $T$ be self-maps of a $p$-normed space $X$ such that $T(\partial C \cap C) \subset C$. Suppose that $f$ and $T$ are commuting on $D = P_C(x_0) \cap D_T^f(x_0)$, $f$ is continuous on $D$, $T$ is $f$-nonexpansive on $D \cup \{x_0\}$ and $f(D) = D$. Suppose that $D$ is compact and has a contractive jointly continuous family $\mathcal{F} = \{h_x\}_{x \in D}$ such that $f(h_x(\alpha)) = h_f(x)(\alpha)$ for all $x \in D$ and all $\alpha \in [0, 1]$. Then, $P_C(x_0) \cap F(f) \cap F(T) \neq \varnothing$. 
We observe that if \( f(P_C(x_0)) = P_C(x_0) = g(P_C(x_0)) \), then \( P_C(x_0) = D^p_C(x_0) \); consequently, Theorem 2.2 is true for \( D = P_C(x_0) \) and hence we have the following extension of the result due to Hicks and Humprishi [5] and Sahab, Khan and Sessa [12].

**Theorem 2.4.** Let \( f \) and \( T \) be self maps on \( X \), \( x_0 \in F(T) \cap F(f) \) and \( C \) be a subset of \( X \) such that \( T(\partial C \cap C) \subset C \). Suppose that \( f \) and \( T \) are commuting on \( D = P_C(x_0) \), \( f \) is continuous on \( D \), \( T \) is \( f \)-nonexpansive on \( D \) \( \cup \{ x_0 \} \) and \( f(D) = D \). Suppose that \( D \) is compact and has a contractive jointly continuous family \( \mathcal{G} = \{ h_x \}_{x \in D} \) such that \( f(h_x(\alpha)) = h_f(x)(\alpha) \) for all \( x \in D \) and \( \alpha \in [0,1] \). Then \( P_C(x_0) \cap F(f) \cap F(T) \neq \emptyset \).

Recall that dual space \( X^* \) (the dual of \( X \)) separates points of \( X \) if for each nonzero \( x \in X \), there exists \( \phi \in X^* \) such that \( \phi(x) \neq 0 \). In this case the weak topology on \( X \) is well-defined. Notice that if \( X \) is not locally convex space, then \( X^* \) need not separate the points of \( X \). For example, if \( X = L^p[0,1], \) \( 0 < p < 1 \) then \( X^* = \{0\} \). However there are some non-locally convex spaces (such as the \( p \)-normed spaces \( L^p, \) \( 0 < p < 1 \) whose dual separates the points.

In the following result we consider \( X \) a complete \( p \)-normed space whose dual separates the points of \( X \).

**Theorem 2.5.** Let \( T \) and \( f \) be selfmaps on \( X \), \( C \) a subset of \( X \) such that \( T(\partial C \cap C) \subset C \) and \( x_0 \in F(T) \cap F(f) \). Let \( D = P_C(x_0) \cap D^p_C(x_0) \) be a nonempty weakly compact set. Suppose that \( f \) is continuous in the weak and the strong topologies on \( D \), \( f(D) = D \), and \( T \) is \( f \)-nonexpansive map on \( D \cup \{ x_0 \} \) which commutes with \( f \) on \( D \). If \( D \) has a contractive family of functions \( \mathcal{G} = \{ h_x \}_{x \in D} \) such that \( f(h_x(\alpha)) = h_f(x)(\alpha) \) for all \( x \in D \) and all \( \alpha \in [0,1] \), then \( f \) and \( T \) have a common fixed point in \( D \) provided either (i) \( T \) is weakly continuous and family \( \mathcal{G} \) is jointly weak continuous or (ii) \( T \) is completely continuous and \( \mathcal{G} \) is jointly continuous.

**Proof.** As in the proof of Theorem 2.2, \( T \) maps \( D \) into \( D \). Define the maps \( T_n \) as in Theorem 2.2. Then \( T_n \) commutes with \( f \) and \( T_n(D) \subset D = f(D) \). Since the family \( \mathcal{G} \) is contractive and \( T \) is \( f \)-nonexpansive we have

\[
\|T_n x - T_n y\|_p \leq \|\phi(\lambda_n)\|^p \|T x - T y\|_p \leq \|\phi(\lambda_n)\|^p \|f x - f y\|_p \quad \text{for all } x, y \in D.
\]

So, by Theorem 5 [6], there exists a unique \( x_n \in D \) such that \( x_n = T_n x_n = f x_n \) for each \( n \). Since \( D \) is weakly compact, there is a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) converging weakly to some \( p \in D \). But, \( f \) is weakly continuous so we have \( f p = p \). Now if (i) holds, then \( T x_{n_j} \to Tp \) and hence \( x_{n_j} = h_{T(x_{n_j})}(\lambda_{n_j}) \to h_{T(p)}(1) = T(p) \). Also since \( x_{n_j} \to p \) and the weak topology is Hausdorff, we get \( Tp = p \). Now suppose the condition (ii) holds. Since \( x_{n_j} \to p \) and \( T \) is completely continuous, we have \( T x_{n_j} \to Tp \). Now using the joint continuity of \( \mathcal{G} \) we get

\[
x_{n_j} = h_{T(x_{n_j})}(\lambda_{n_j}) \to h_{T(p)}(1) = T(p).
\]

Thus \( T x_{n_j} \to T^2 p \) and consequently \( Tz = z \) where \( z = Tp \). But

\[
fz = fTp = Tfp = Tp = z,
\]
which completes the proof.

References


Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran-31261, Saudi Arabia.
E-mail: arahim@kfupm.edu.sa

Department of Mathematics, King Abdul Aziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia.
E-mail: Latifmath@yahoo.com

Department of Mathematics, Gomal University, D. I. Khan, Pakistan.
E-mail: arjamandbano2002@yahoo.com

Department of Mathematics, King Abdul Aziz University, P.O.Box 80203, Jeddah 21589, Saudi Arabia.
E-mail: mnawab2000@yahoo.com