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SOME RESULTS ON COMMON FIXED POINTS AND BEST APPROXIMATION

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Abstract. We prove a common fixed point result for (f, g)-nonexpansive maps and then derive certain results on best approximation. Our results generalize the results of Al-Thagafi [1], Jungck and Sessa [6], Khan, Hussain and Thaheem [8], Latif [9, 10] and Sahab, Khan and Sessa [12].

1. Preliminaries.

Let X be a linear space. A p-norm on X is a real valued function $\|.\|_p$ on X with $0 < p \le 1$, satisfying the following conditions:

(i) $\|x\|_p \ge 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$

(ii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$

(iii) $\|x + y\|_p \le \|x\|_p + \|y\|_p$

for all x, $y \in X$ and all scalars λ. The pair $(X, \|.\|_p)$ is called a p-normed space. It is a metric space with a translation-invariant metric d_p defined by $d_p(x, y) = \|x - y\|_p$ for all x, $y \in X$. If $p = 1$, we obtain the concept of a normed space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \le 1$ [9, 10]. The space l_p and L_p, $0 < p \le 1$ are p-normed spaces. A p-normed space is not necessarily a locally convex space.

Let C be a subset of p-normed space X, and let f, g be self maps of X. The set C is called starshaped with respect to $q \in C$ if for all $x \in C$ and for all k, $0 \le k \le 1$, $kx + (1 - k)q \in C$. The set C is said to be starshaped if it is starshaped with respect to one of its elements.

Each convex set is necessarily starshaped. For any $x_0 \in X$, define $d_p(x_0, C) = \inf_{u \in C} \|x_0 - u\|_p$, $P_C(x_0) = \{y \in C : \|y - x_0\|_p = d_p(x_0, C)\}$ the set of best C-approximants to x_0 and $D_C^f(x_0) = \{y \in C : f(y) \in P_C(x_0)\}$. Note that $P_C(x_0)$ contains $f(D_C^f(x_0))$ and $g(D_C^g(x_0))$. Assume that $D_C^{f,g}(x_0) = D_C^f(x_0) \cap D_C^g(x_0)$ and $D = P_C(x_0) \cap D_C^{f,g}(x_0)$.

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Let $\Im = \{h_\alpha\}_{\alpha \in C}$ be a family of functions from $[0,1]$ into C such that $h_\alpha(1) = \alpha$ for each $\alpha \in C$. The family $\Im$ is said to be contractive if there exists a function $\varphi : (0,1) \to (0,1)$ such that for all α, $\beta \in C$ and all $t \in (0,1)$ we have

$$\|h_\alpha(t) - h_\beta(t)\|_p \leq [\varphi(t)]^p \|\alpha - \beta\|_p$$

The family $\Im$ is said to be jointly continuous (resp. jointly weakly continous) if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in C (resp. $t \to t_0$ in $[0,1]$ and $\alpha \xrightarrow{w} a_0$ in C), then $h_\alpha(t) \to h_{\alpha_0}(t_0)$ (resp. $h_\alpha(t) \xrightarrow{w} h_{\alpha_0}(t)$) in C; here $\to$ and $\xrightarrow{w}$ denote the strong and weak convergence respectively.

It is known that if $C \subseteq X$ is q-starshaped and $h_x(t) = (1-t)q + tx$, $(x \in C; t \in [0,1])$, then $\Im = \{h_x\}_{x \in C}$ is a contractive jointly continuous family with $\varphi(t) = t$. Thus the class of subsets of X with the property of contractiveness and jointly continuity contains the class of starshaped sets which in turns contains set class of convex sets (see, [3] and [8]). If for a subset C of X, there exists a contractive jointly continuous family of functions $\Im = \{h_\alpha\}_{\alpha \in C}$, then we say that C has the property of contractiveness and joint continuity.

A map $T : X \to X$ is (f,g)-contraction [7, 9], if there exists a real number $k \in (0,1)$ such that

$$\|Tx - Ty\|_p \leq k\|fx - gy\|)p \quad \text{for all } x, y \in X.$$

If in the above inequality $k = 1$, then T is called (f,g)-nonexpansive. Also if $f = g$, we say that T is f-nonexpansive. We denote the boundary of C by ∂C, closure of C by $cl(C)$ and the set of fixed points of T by $F(T)$.

Using fixed point theory, Brosowski [2], Meinardus [11] have established some interesting results on invariant approximation in the setting of normed spaces. Jungck and Sessa [6] have also obtained some results in approximation theory in the setting of normed spaces. Their work has been extended, generalized and unified by many authors; for example, see [4, 5, 6, 8, 12, 13].

Recently, Latif [9] has obtained the following results on common fixed points and best approximation, which generalize and extend the recent work of Al-Thagafi [1], Sahab, Khan and Sessa [12] and Kaneko [7] etc.

Theorem 1.1. *Let X be a p-normed space and C a closed subset of X which is starshaped with respect to q. Let f, g and T be continuous self-maps of C such that $T(C) \subset f(C) \cap g(C)$, T commutes with f and g, and $q \in F(f) \cap F(g)$. If $cl(T(C))$ is compact, f and g are affine, and T is (f,g)-nonexpansive, then $F(f) \cap F(g) \cap F(T) \neq \varnothing$.*

Theorem 1.2. *Let f, g and T be self-maps of a p-normed space X such that $x_0 \in F(f) \cap F(g) \cap F(T)$ and $C \subset X$ with $T(\partial C \cap C) \subset C$. Let T be a (f,g)-nonexpansive map and continuous on $D \cup \{x_0\}$ such that $cl(T(D))$ is compact, and let f and g be continuous, surjective, affine and commute with T on D. If D is closed and starshaped with respect to $q \in F(f) \cap F(g)$, then $P_C(x_0) \cap F(f) \cap F(g) \cap F(T) \neq \varnothing$.*

We also recall the following common fixed point result due to Jungck and sessa [6].

Theorem 1.3. *Let f be a continuous self-map of a compact metric space (X, d). If $fx \neq fy$ implies $d(fx, fy) < d(gx, hy)$ for some self-maps g and h of X, commuting with f, then $F(f) \cap F(g) \cap F(h)$ is singleton.*

The aim of this paper is to extend Theorem 1.1 and 1.2 to a domain which is not necessarily starshaped. An approximation result for two maps when underlying best approximation set D need not be compact, will be established. Our results extend the corresponding results of Jungck and Sessa [6], Khan, Hussain and Thaheem [8], Latif [9, 10], Sahab, Khan and Sessa [12], Habiniak [4] and Singh [13].

2. Common Fixed Points and Best Approximations

Our first result generalizes Theorem 1.1, in the sense that the underlying domain C need not be starshaped.

Theorem 2.1. *Let f, g and T be continuous self-maps on a subset C of a p-normed space X. Suppose that T is (f, g)-nonexpansive and commutes with f and g on C. Suppose that C is compact and has a contractive jointly continuous family $\Im = \{h_x\}_{x \in C}$ such that $f(h_x(\alpha)) = h_{f(x)}(\alpha)$ and $g(h_x(\alpha)) = h_{g(x)}(\alpha)$ for all $x \in C$ and $\alpha \in [0, 1]$. Then $F(f) \cap F(g) \cap F(T) \neq \varnothing$.*

Proof. For $n \in N$, let $\lambda_n = n(n+1)^{-1}$. Define

$$T_n(x) = h_{(Tx)}(\lambda_n), \quad x \in C.$$

Then each T_n is a well-defined self-map of C. Since f and T commute on C, it follows from the property of $\Im$ that

$$T_n(fx) = h_{Tf(x)}(\lambda_n) = h_{fT(x)}(\lambda_n) = f(h_{T(x)}(\lambda_n)) = fT_n(x) \quad (x \in C).$$

Thus for each n, T_n commutes with f. Similarly, for each n, T_n commutes with g.

Moreover, since T is (f, g)-nonexpansive, we get

$$\begin{aligned} \|T_n x - T_n y\|_p &= \|h_{T(x)}(\lambda_n) - h_{T(y)}(\lambda_n)\|_p \\ &\leq [\varphi(\lambda_n)]^p \|T(x) - T(y)\|_p \\ &\leq [\varphi(\lambda_n)]^p \|fx - gy\|_p \\ &< \|fx - gy\|_p \end{aligned}$$

Then it follows from Theorem 1.3 that there exists $x_n \in C$ such that $x_n \in F(T_n) \cap F(f) \cap F(g)$ for each n. Thus, by the compactness of C it follows that the sequence, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to some $z \in C$ and hence $T(x_{n_i}) \to T(z)$. The joint continuity of $\Im$ gives

$$x_{n_i} = T_{n_i}(x_{n_i}) = h_{T(x_{n_i})}(\lambda_{n_i}) \to h_{T(z)}(1) = Tz$$

and thus the uniqueness of the limit implies $Tz = z$. Also, since $fx_{n_i} = x_{n_i} = gx_{n_i} \to z$ using the continuity of f and g and the uniqueness of the limit, we have $f(z) = g(z) = z$, which completes the proof.

Theorem 2.2. *Let f, g and T be self-maps of a p-normed space X such that $x_0 \in F(f) \cap F(g) \cap F(T)$ and $C \subset X$ with $T(\partial C \cap C) \subset C$. Let T is (f, g)-nonexpansive on $D \cup \{x_0\}$ and let f and g be continuous, surjective and commute with T on D. Suppose that D is compact and has a contractive jointly continuous family $\Im = \{h_x\}_{x \in D}$ such that $f(h_x(\alpha)) = h_{f(x)}(\alpha)$ for all $x \in D$ and $\alpha \in [0, 1]$. Then $P_C(x_0) \cap F(f) \cap F(g) \cap F(T) \neq \varnothing$.*

Proof. Let $y \in D$. Then $\|y - x_0\|_p = d_p(x_0, C)$ since $y \in P_C(x_0)$. Note that for any $k \in (0, 1)$

$$\|kx_0 + (1 - k)y + x_0\|_p = (1 - k)^p \|y - x_0\|_p < d_p(x_0, C)$$

It follows that the line segment $\{kx_0 + (1 - k)y : 0 < k < 1\}$ and the set C are disjoint. Thus y is not in the interior of C and so $y \in \partial C \cap C$. Since $T(\partial C \cap C) \subset C$, Ty must be in C. Also since $fy \in P_C(x_0)$, $x_0 \in F(T) \cap F(g)$ and T is (f, g)-nonexpansive on $D \cup \{x_0\}$, we have

$$\|Ty - x_0\|_p = \|Ty - Tx_0\|_p \leq \|fy - gx_0\|_p = \|fy - x_0\|_p = d_p(x_0, C),$$

and hence $Ty \in P_C(x_0)$. Moreover, since T commutes with f on D and $x_0 \in F(g) \cap F(T)$, we have

$$\|fTy - x_0\|_p = \|Tfy - Tx_0\|_p \leq \|f^2y - gx_0\|_p = \|f^2y - x_0\| = d_p(x_0, C)$$

and similarly, since T commutes with g and $x_0 \in F(f) \cap F(T)$, we get

$$\|gTy - x_0\|_p \leq \|g^2y - x_0\|_p = d_p(x_0, C).$$

Thus fTy and gTy are in $P_C(x_0)$ and so $Ty \in D_C^{f,g}(x_0)$. Thus the definition of D implies that $Ty \in D$. Consequently T maps D into D. For $n \in N$, define the maps T_n from D into D as in the proof of Theorem 2.1. As in the proof of Theorem 2.1, there exists $x_n \in D$ such that $x_n \in F(T_n) \cap F(f) \cap F(g)$ for each n. In particular $x_n = fx_n = gx_n$, by compactness of D, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges to $z \in D$ and hence $T(x_{n_i}) \to Tz$. Now, we complete the proof as that of Theorem 2.1.

Now if we take $f = g$ in Theorem 2.2, then $D = P_C(x_0) \cap D_C^f(x_0)$ and thus we obtain the following result which generalizes Theorem 3.2 of Al-Thagafi [1] in the sense that the best approximation set D in it need not be starshaped.

Corollary 2.3. *Let f, T be self-maps of a p-normed space X such that $T(\partial C \cap C) \subset C$. Suppose that f and T are commuting on $D = P_C(x_0) \cap D_C^f(x_0)$, f is continuous on D, T is f-nonexpansive on $D \cup \{x_0\}$ and $f(D) = D$. Suppose that D is compact and has a contractive jointly continuous family $\Im = \{h_x\}_{x \in D}$ such that $f(h_x(\alpha)) = h_{f(x)}(\alpha)$ for all $x \in D$ and all $\alpha \in [0, 1]$. Then, $P_C(x_0) \cap F(f) \cap F(T) \neq \varnothing$.*

We observe that if $f(P_C(x_0)) = P_C(x_0) = g(P_C(x_0))$, then $P_C(x_0) = D_C^{f,g}(x_0)$; consequently Theorem 2.2 is true for $D = P_C(x_0)$ and hence we have the following extension of the result due to Hicks and Humprise [5] and Sahab, Khan and Sessa [12].

Theorem 2.4. *Let f and T be self maps on X, $x_0 \in F(T) \cap F(f)$ and C be a subset of X such that $T(\partial C \cap C) \subset C$. Suppose that f and T are commuting on $D = P_C(x_0)$, f is continuous on D, T is f-nonexpansive on $D \cup \{x_0\}$ and $f(D) = D$. Suppose that D is compact and has a contractive jointly continuous family $\Im = \{h_x\}_{x \in D}$ such that $f(h_x(\alpha)) = h_{f(x)}(\alpha)$ for all $x \in D$ and $\alpha \in [0,1]$. Then $P_C(x_0) \cap F(f) \cap F(T) \neq \varnothing$.*

Recall that dual space X^* (the dual of X) separates points of X if for each nonzero $x \in X$, there exists $\phi \in X^*$ such that $\phi(x) \neq 0$. In this case the weak topology on X is well-defined. Notice that if X is not locally convex space, then X^* need not separate the points of X. For example, if $X = L_p[0,1]$, $0 < p < 1$, then $X^* = \{0\}$. However there are some non-locally convex spaces (such as the p-normed spaces L_p, $0 < p < 1$) whose dual separates the points.

In the following result we consider X a complete p-normed space whose dual separates the points of X.

Theorem 2.5. *Let T and f be selfmaps on X, C a subset of X such that $T(\partial C \cap C) \subset C$ and $x_0 \in F(T) \cap F(f)$. Let $D = P_C(x_0) \cap D_C^f(x_0)$ be a nonempty weakly compact set. Suppose that f is continuous in the weak and the strong topologies on D, $f(D) = D$, and T is f-nonexpansive map on $D \cup \{x_0\}$ which commutes with f on D. If D has a contractive family of functions $\Im = \{h_x\}_{x \in D}$ such that $f(h_x(\alpha)) = h_{f(x)}(\alpha)$ for all $x \in D$ and all $\alpha \in [0,1]$, then f and T have a common fixed point in D provided either (i) T is weakly continuous and family $\Im$ is jointly weak continuous or (ii) T is completely continous and $\Im$ is jointly continuous.*

Proof. As in the proof of Theorem 2.2, T maps D into D. Define the maps T_n as in Theorem 2.2. Then T_n commutes with f and $T_n(D) \subseteq D = f(D)$. Since the family $\Im$ is contractive and T is f-nonexpansive we have

$$\|T_n x - T_n y\|_p \leq [\phi(\lambda_n)]^p \|Tx - Ty\|_p \leq [\phi(\lambda_n)]^p \|fx - fy\|_p \quad \text{for all } x, y \in D.$$

So, by Theorem 5 [6], there exists a unique $x_n \in D$ such that $x_n = T_n x_n = f x_n$ for each n. Since D is weakly compact, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $p \in D$. But, f is weakly continuous so we have $fp = p$. Now if (i) holds, then $Tx_{n_j} \xrightarrow{w} Tp$ and hence $x_{n_j} = h_{T(x_{n_j})}(\lambda_{n_j}) \xrightarrow{w} h_{T(p)}(1) = T(p)$. Also since $x_{n_j} \xrightarrow{w} p$ and the weak topology is Hausdorff, we get $Tp = p$. Now suppose the condition (ii) holds. Since $x_{n_j} \xrightarrow{w} p$ and T is completely continuous, we have $Tx_{n_j} \to Tp$. Now using the joint continuity of $\Im$ we get

$$x_{n_j} = h_{T(x_{n_j})}(\lambda_{n_j}) \to h_{T(p)}(1) = T(p).$$

Thus $Tx_{n_j} \to T^2 p$ and consequently $Tz = z$ where $z = Tp$. But

$$fz = fTp = Tfp = Tp = z,$$

which completes the proof.

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