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UNIFIED PRESENTATION OF CERTAIN FAMILIES OF ELLIPTIC-TYPE INTEGRALS RELATED TO EULER INTEGRALS AND GENERATING FUNCTIONS

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Abstract. The aim of the present paper is to give unified presentation of certain families of elliptic-type integrals related to Euler integrals and generating functions. It is a unification and generalization of certain families of elliptic-type integrals which were studied in a number of earlier works on the subject due to their importance and potential in certain problems in radiation physics and nuclear technology. The results established in this paper are of manifold generality and basic in nature. Besides deriving various known and new elliptic-type integrals and their generalizations these theorems can be used to evaluate various Euler-type integrals involving a number of generating functions.

1. Introduction

$$\Omega_j(k) = \int_0^\pi (1 - k^2 \cos\theta)^{-j-\frac{1}{2}} d\theta, \tag{1}$$

where $j = 0, 1, 2, \ldots$ and $0 \le k < 1$ was studied by Epstein-Hubbel [5], for the first time. Due to its occurrence in a number of physical problems [1, 2, 6, 8, 15, 17, 29], in the form of single and multiple integrals, several authors notably Kalla [9, 10] and Kalla et al. [11], Kalla and Al-Saqabi [12], Kalla et al. [13], Salman [23], Saxena et al. [26], Srivastava and Bromberg [31], have investigated various interesting unifications (and generalizations) of the elliptic-type integrals (1). Some of the important generalizations of elliptic-type integrals (1) are as follows. Kalla [9, 10] introduced the generalization of the form:

$$R_\mu(k, \alpha, \gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\gamma-2\alpha-1}(\theta/2)}{(1 - k^2 \cos\theta)^{\mu+\frac{1}{2}}} d\theta, \tag{2}$$

where $0 \le k < 1$, $\operatorname{Re}(\gamma) > \operatorname{Re}(\alpha) > O$, $\operatorname{Re}(\mu) > -\frac{1}{2}$.

Results for this generalization are also derived by Glasser and Kalla [7]. Al-Saqabi [18] defined and studied the generalization given by the integral

$$B_\mu(k, m, \nu) = \int_0^\pi \frac{\cos^{2m}(\theta) \sin^{2\nu}(\theta)}{(1 - k^2 \cos\theta)^{\mu+\frac{1}{2}}} d\theta, \tag{3}$$

where $0 \le k < l$, $m \in N_0$, $\mu \in C$, $\mathrm{Re}\,(\mu) > -\frac{1}{2}$.

Asymptotic expansion of (3) has recently been discussed by Matera et al. [20]. The integral

$$\Lambda_\nu(\alpha, k) = \int_0^\pi \frac{\exp[\alpha \sin^2(\theta/2)]}{(1 - k^2 \cos\theta)^{\nu + \frac{1}{2}}} d\theta, \tag{4}$$

where $0 \le k < 1$, $\alpha, \nu \in R$; presents another generalization of (1), given by Siddiqi [29].

Srivastava and Siddiqi [35] have given an interesting unification and extension of the families of elliptic-type integrals in the following form:

$$\Lambda_{(\lambda,\mu)}^{(\alpha,\beta)}(\rho; k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1 - k^2 \cos\theta)^{\mu + \frac{1}{2}}}[1 - \rho \sin^2(\theta/2)]^{-\lambda} d\theta, \tag{5}$$

where $0 \le k < 1$, $\mathrm{Re}\,(\alpha) > O$, $\mathrm{Re}\,(\beta) > 0$, $\lambda, \mu \in C$, $|\rho| < 1$.

Kalla and Tuan [14] generalized equation (5) by means of the following integral and also obtained its asymptotic expansion

$$\Lambda_{(\lambda,\gamma,\mu)}^{(\alpha,\beta)}(\rho, \delta; k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)}{(1 - k^2 \cos\theta)^{\mu + \frac{1}{2}}}[1 - \rho \sin^2(\theta/2)]^{-\lambda}[1 + \delta \cos^2(\theta/2)] d\theta, \tag{6}$$

where $0 \le k < 1$, $\mathrm{Re}\,(\alpha) > 0$, $\mathrm{Re}\,(\beta) > 0$, $\lambda, \mu, \gamma \in C$ and either $|\rho|$, $|\delta| < 1$ or p (or S) $\in C$ whenever $\lambda = m$ or $\gamma = -m$, $m \in N_0$, respectively.

Al-Zamel et al. [19] discussed a generalized family of elliptic-type integrals in the form:

$$\begin{aligned}
Z_{(\gamma)}^{(\alpha,\beta)}(k) &= Z_{(\gamma_1,\dots,\gamma_n)}^{(\alpha,\beta)}(k_1,\dots,k_n) \\
&= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2) \prod_{j=1}^n (1 - k_j^2 \cos\theta)^{-\gamma_j} d\theta \\
&= B(\alpha, \beta) \prod_{j=1}^n (1 - k_j^2)^{-\gamma_j} F_D^{(n)}\Big(\beta; \gamma_1,\dots,\gamma_n; \frac{2k_1^2}{k_1^2 - 1},\dots,\frac{2k_n^2}{k_n^2 - 1}\Big),
\end{aligned} \tag{7}$$

where $\mathrm{Re}\,(\alpha) > O$, $\mathrm{Re}\,(\beta) > O$, $|k_j| < 1$; $\gamma_j \in C(j = 1,\dots,n)$ $F_D^{(n)}(\cdot)$ is the Lauricella hypergeometric function of n-variables [21, p.163].

Saxena and Kalla [27] have studied a family of elliptic-type integrals of the form:

$$\begin{aligned}
&\Omega_{(\sigma_1,\dots,\sigma_{n-2};\delta,\mu)}^{(\alpha,\beta)}(\rho_1,\dots,\rho_{n-2},\delta; k) \\
&= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2) \prod_{j=1}^{(n-2)} \frac{[1 - \rho_j \sin^2(\theta/2)]^{-\sigma_j}[1 + \delta \cos^2(\theta/2)]^{-\gamma}}{(1 - k^2 \cos\theta)^{\mu + \frac{1}{2}}} d\theta, \tag{8}
\end{aligned}$$

where $0 \le k < 1$, $\mathrm{Re}\,(\alpha) > O$, $\mathrm{Re}\,(\beta) > O$; σ_j $(j = 1,\dots,n-2)$, $\gamma, \mu \in C$;

$$\max\Big\{|\rho_j|, \Big|\frac{\delta}{1+\delta}\Big|, \Big|\frac{2k^2}{k^2 - 1}\Big|\Big\} < 1.$$

In a recent paper, Saxena and Pathan [28] investigated an extension of equation (8) in the form:

$$
\Omega_{(\sigma_1,\ldots,\sigma_\mu;\gamma;L_1,\ldots,L_n)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_\mu,\delta;\lambda_1,\ldots,\lambda_n)
$$

$$
= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)\prod_{i=1}^m [1-\rho_i\sin^2(\theta/2)]^{-\sigma_i}
$$

$$
\cdot[1+\delta\cos^2(\theta/2)]^{-\gamma}\prod_{j=1}^n (1-\lambda_j^2\cos\theta)^{-L_j}\,d\theta, \tag{9}
$$

where $\min\{\mathrm{Re}\,(\alpha),\mathrm{Re}\,(\beta)\} > 0$; $|\lambda_j| < 1$; $\sigma_i,\gamma,L_j \in C$;

$$
\max\left\{|\rho_i|,\ \left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|,\ \left|\frac{\delta}{1+\delta}\right|\right\} < 1\ (i=1,\ldots,m;\ j=1,\ldots,n).
$$

In a recent paper [4], Chaurasia and Pandey investigated a new family of unified and generalized elliptic-type integrals:

$$
\overline{\Omega}_{(\lambda_i,L_i)}^{(\alpha,\beta)}((\rho_i),(\delta_i);k_j) = \overline{\Omega}_{(\lambda_1,\ldots,\lambda_n;L_1,\ldots,L_M)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_N,\delta_1,\ldots,\delta_N;k_1,\ldots,k_M)
$$

$$
= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)\prod_{i=1}^m [1+\rho_i\sin^2(\theta/2)+\delta_i\cos^2(\theta/2)]^{-\lambda_i}\prod_{j=1}^n (1-k_j^2\cos\theta)^{-L_j}\,d\theta, \tag{10}
$$

where $\min\{\mathrm{Re}\,(\alpha),\mathrm{Re}\,(\beta)\} > 0$; $|k_j| < 1$; $\lambda_i,L_i \in C$;

$$
\max\left\{|\rho_i|,\ |\delta_i|,\ \left|\frac{2k_j^2}{k_j^2-1}\right|,\ \left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|\right\} < 1\ (i=1,\ldots,N;\ j=1,\ldots,M)
$$

which includes most of the known generalized and unified families of elliptic-type integrals (including those discussed in (1) through (9)). For more details also see [10], [25] and [29].

Upon a closer examination of the above equation (10), it can be seen that the family of elliptic-type integral $\overline{\Omega}_{(\lambda_i,L_j)}^{(\alpha,\beta)}((\rho_i),(\delta_i);k_j)$ can be put in to the following form involving Euler-type integral:

$$
\overline{\Omega}_{(\lambda_1,\ldots,\lambda_N;L_1,\ldots,L_M)}^{(\alpha,\beta)}(\rho_1,\ldots,\rho_N,\delta_1,\ldots,\delta_N;k_1,\ldots,k_M)
$$

$$
= \prod_{j=1}^M (1-k_j^2)^{-L_j}\prod_{i=1}^N (1+\delta_i)^{-\lambda_i}\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\prod_{j=1}^M \left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-L_j}\prod_{i=1}^N \left[1-\frac{(\delta_i-\rho_i)\omega}{(1+\delta_i)}\right]^{-\lambda_i}d\omega. \tag{11}
$$

A two-variable generating function $F(x,t)$ possess a formal (not necessarily convergent for $t \neq 0$) power series representation in t, can be written in the following form

$$
F(x,t) = \sum_{n=0}^\infty C_n f_n(x)t^n, \tag{12}
$$

where each member of the generalized set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of x and t. The well known H-function [32] is defined in the form

$$H_{P_\ell,Q_\ell}^{M_\ell,N_\ell}\left[z\,\middle|\,\begin{matrix}(a_j,A_j)_{1,P_\ell}\\(b_j,B_j)_{1,Q_\ell}\end{matrix}\right] = \frac{1}{2\pi i}\int_E \theta(s)z^s ds, \tag{13}$$

where

$$\theta(s) = \frac{\displaystyle\prod_{j=1}^{M_\ell}\Gamma(b_j - B_j s)\prod_{j=1}^{N_\ell}\Gamma(1 - a_j + A_j s)}{\displaystyle\prod_{j=M_\ell+1}^{Q_\ell}\Gamma(1 - b_j + B_j s)\prod_{j=N_\ell+1}^{P_\ell}\Gamma(a_j - A_j s)}.$$

$\ell = 1,\ldots,r$, $i = \sqrt{-1}$, z is not equal to zero but may be real or complex number and an empty product is interpreted as unity. $M_\ell, N_\ell, P_\ell, Q_\ell$ are non-negative integers satisfying $0 \le N_\ell \le P_\ell$, $O \le M_\ell \le Q_\ell$. The parameters $A_1,\ldots,A_{P_\ell}$; $B_1,\ldots,B_{Q_\ell}$, are real positive numbers, $a_1,\ldots,a_{P_\ell}$; $b_1,\ldots,b_{Q_\ell}$ are complex numbers such that the poles of the Gamma functions in the integrand in (13) do not coincide. E is a suitable contour in the s-plane separating the poles of Gamma products with $+s$ and $-s$ in the numerator.

The H-function of several complex variables introduced by H. M. Srivastava and Rekha Panda [33, p.265] is represented and defined in the following manner:

$$H_{0,\lambda:(u',v');\ldots;(u^r,v^r)}^{A,C:[B',D'];\ldots;[B^r,D^r]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^r]:[b';\phi'];\ldots;[b^r;\phi^r];\\ [(c):\psi',\ldots,\psi^r]:[d';\delta'];\ldots;[d^r,\delta^r];\end{matrix}\,Z_1,\ldots,Z_r\right]$$
$$= \frac{1}{(2\pi i)^r}\int_{E_1}\cdots\int_{E_r} R_1(s_1)\cdots R_r s_r\, T(s_1,\ldots,s_r)z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \tag{14}$$

where

$$R_i(s_i) = \frac{\displaystyle\prod_{j=1}^{u^{(i)}}\Gamma(d_j^i - \delta_j^i s_i)\prod_{j=1}^{v^i}\Gamma(1 - b_j^i + \phi_j^i s_i)}{\displaystyle\prod_{j=u^i+1}^{D^i}\Gamma(1 - d_j^i + \delta_j^i s_i)\prod_{j=v^i+1}^{B^i}\Gamma(b_j^i - \phi_j^i s_i)}, \qquad \forall\ i = 1,\ldots,r \tag{15}$$

$$T(s_1,\ldots,s_r) = \frac{\displaystyle\prod_{j=1}^{\lambda}\Gamma(1 - a_j + \sum_{i=1}^{r}\theta_j^i s_i)}{\displaystyle\prod_{j=\lambda+1}^{A}\Gamma(a_j - \sum_{i=1}^{r}\theta_j^i s_i)\prod_{j=1}^{C}\Gamma(1 - c_j + \sum_{i=1}^{r}\psi_j^i s_i)} \tag{16}$$

$i = \sqrt{-1}$, an empty product is interpreted as unity, the coefficients θ_j^i, $j = 1,\ldots,A$; ϕ_j^i, $j = 1,\ldots,B^i$; ψ_j^i, $j = 1,\ldots,C$; δ_j^i, $j = 1,\ldots,D^i$ $\forall i = 1,\ldots,r$ are positive numbers and $\lambda, u^i, v^i, A, B^i$, C, D^i are integers such that $0 \le \lambda \le A$, $1 \le u^i \le D^i$, $C \ge 0$ and $0 \le V^i \le B^i$ $\forall\ i = 1,\ldots,r$. The

contour E_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-i\infty$ to $+i\infty$ with indentations, if necessary, in such a manner that all poles of $\Gamma(d_j^i - \delta_j^i s_i)$, $j = 1,\ldots,u^i$, are to the right and those of $\Gamma(1 - b_j^i + \phi_j^i s_i)$, $j = 1,\ldots,v^i$ and $\Gamma(1 - a_j + \sum_{i=1}^r \theta_j^i s_i)$, $j = 1,\ldots,\lambda$, to the left of s_i, the various parameters being so restricted that these poles are all simple and none of them coincide and with the points z_i; $i = 1,\ldots,r$ being tacitly excluded. When A_j $(j = 1,\ldots,P_\ell) = B_J(j = 1,\ldots,Q_\ell) = 1$ in (13), it reduces to a Meijer's G-function [32].

If we take $\theta',\ldots,\theta^r = \phi',\ldots,\phi^r = \psi',\ldots,\psi^r = \delta',\ldots,\delta^r = 1$ in (14), we have the following interesting transformation:

$$H_{A,C:[B',D'];\ldots;[B^r,D^r]}^{0,\lambda:(u',v');\ldots;(u^r,v^r)} \left[\begin{array}{c} [(a):1,\ldots,1]:[b';1];\ldots;[b^r;1]; \\ {[(c):1,\ldots,1]:[d';1];\ldots;[d^r,1];} \end{array} Z_1,\ldots,Z_r \right]$$

$$= G_{A,C:[B',D'];\ldots;[B^r,D^r]}^{0,\lambda:(u',v');\ldots;(u^r,v^r)} \left[\begin{array}{c} [(a):b,\ldots,b^r]; \\ {[(c):d',\ldots,d^r];} \end{array} Z_1,\ldots,Z_r \right]. \tag{17}$$

In this article we have studied two new theorems associated with two variables generating functions. Beside generalizing most of the known elliptic-type integrals some new families of elliptio-type integrals can also be deduced with the help of the results obtained in this article. Such generalized and new families of elliptictype integrals play an important role in evaluation of a number of Euler-type integrals involving various generating functions. The basic idea of evolving the theorems discussed in this article is inspired by the research work of Mohammed [22], Saran [24] and Srivastava and Yeh [36].

2. Theorems

In this section we derive two new theorems and their corollaries on generating functions associated with the families of elliptic-type integrals. These theorems and corollaries can be used to establish various known and new elliptic type integrals. Some of the significant applications of the results derived in this section are discussed in the Section 3.

Theorem 1. *Let the generating function $F(x,t)$ and $H_{P_\ell,Q_\ell}^{M_\ell,N_\ell}(z)$ be the H-function of one variable defined in* (12) *and* (13) *respectively. Then*

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^r H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell)^{\xi_\ell} \left| \begin{array}{c} (a_j^\ell, A_j^\ell)_{1,P_\ell} \\ (b_j^\ell, B_j^\ell)_{1,Q_\ell} \end{array} \right. \right] F[x, t\omega^\eta(1-\omega)^\mu]d\omega$$

$$= \Gamma(\gamma-\alpha) \sum_{n=0}^\infty (\gamma-\alpha)_{\mu n} C_n f_n(x) t^n$$

$$\cdot H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)} \left[\begin{array}{c} [(1-\alpha-\eta n):\xi_1,\ldots,\xi_r]:[a_j';A_j'];\ldots;[a_j^r;A_j^r]; \\ {[(1-\gamma-\eta n-\mu n):\xi_1,\ldots,\xi_r]:[b_j';B_j'];\ldots;[b_j^r;B_j^r];} \end{array} Z_1^{\xi_1},\ldots,Z_r^{\xi_r} \right], \tag{18}$$

provided that $\min\{\text{Re}(\alpha), \text{Re}(\gamma-\alpha)\} > O$, $\text{Re}(\eta) > O$, $\text{Re}(\mu) > 0$ *and* $\max|z_\ell| \to 0$ $(\ell = 1,\ldots,r)$ *and* $\min|z_\ell| \to \infty$ $(\ell = 1\ldots,r)$.

On taking $\xi_\ell = 1$ (for $\ell = 1, \ldots, r$), Theorem 1 gives the following corollary:

Corollary 1. *Let the generating function $F(x, t)$ and $H_{P_\ell, Q_\ell}^{M_\ell, N_\ell}(z)$ be the H-function of one variable defined in* (12) *and* (13) *respectively. Then*

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^r H_{P_\ell, Q_\ell}^{M_\ell, N_\ell}\left[(\omega z_\ell) \left|\begin{array}{c}(a_j^\ell, A_j^\ell)_{1, P_\ell}\\(b_j^\ell, B_j^\ell)_{1, Q_\ell}\end{array}\right.\right] F[x, t\omega^\eta(1-\omega)^\mu] d\omega$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^\infty (\gamma - \alpha)_{\mu n} C_n f_n(x) t^n$$

$$\cdot H_{A+1, C+1:[P_1, Q_1];\ldots;[P_r, Q_r]}^{0, \lambda+1:(M_1, N_1);\ldots;(M_r, N_r)} \left[\begin{array}{c}(1-\alpha-\eta n):[a_j'; A_j'];\ldots;[a_j^r; A_j^r];\\(1-\gamma-\eta n-\mu n):[b_j'; B_j'];\ldots;[b_j^r; B_j^r];\end{array} Z_1, \ldots, Z_r\right], \tag{19}$$

valid under the conditions as given in (18).

On taking $\xi_1 = \cdots = \xi_r = A_j' = \cdots = A_j^r = B_j' = \cdots = B_j^r = \eta = \mu = 1$, Theorem 1 reduces in the following corollary:

Corollary 2. *Let the generating function $F(x, t)$ and $H_{P_\ell, Q_\ell}^{M_\ell, N_\ell}(z)$ be the H-function of one variable defined in* (12) *and* (13) *respectively. Then*

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^r H_{P_\ell, Q_\ell}^{M_\ell, N_\ell}\left[(\omega z_\ell) \left|\begin{array}{c}(a_j^\ell, A_j^\ell)_{1, P_\ell}\\(b_j^\ell, B_j^\ell)_{1, Q_\ell}\end{array}\right.\right] F[x, t\omega(1-\omega)] d\omega$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^\infty (\gamma - \alpha)_n C_n f_n(x) t^n$$

$$\cdot G_{A+1, C+1:[P_1, Q_1];\ldots;[P_r, Q_r]}^{0, \lambda+1:(M_1, N_1);\ldots;(M_r, N_r)} \left[\begin{array}{c}[(1-\alpha-n):a_j'; a_j']\\{[(1-\gamma-2n):b_j', \ldots, \beta_j^r]};\end{array} Z_1, \ldots, Z_r\right], \tag{20}$$

where $G_{A+1, C+1}^{0, \lambda+1}[z_1, \ldots, z_r]$ involved in (20) *is the Meijer's G-function of r-variables [16].*

Now we state an another modification of the Theorem 1, which can be used to obtain various (new or known) interesting generalization of elliptic-type integrals.

Theorem 2. *Let the generating function function $F(x, t)$ and $H_{P_\ell, Q_\ell}^{M_\ell, N_\ell}(z)$ be the H-function of one variable defined in* (12) *and* (13) *respectively. Then*

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{\ell=1}^r H_{P_\ell, Q_\ell}^{M_\ell, N_\ell}\left[(\omega z_\ell)^{\xi_\ell} \left|\begin{array}{c}(a_j^\ell, A_j^\ell)_{1, P_\ell}\\(b_j^\ell, B_j^\ell)_{1, Q_\ell}\end{array}\right.\right]$$

$$\cdot \prod_{j=1}^M \left[1 - \frac{2\omega k_j^2}{k_j^2 - 1}\right]^{-L_j} \prod_{i=1}^N \left[1 - \frac{(\delta_i - \rho_i)\omega}{(1+\delta_i)}\right]^{-\lambda_i} F[x, t\omega^\eta(1-\omega)^\mu] d\omega$$

$$= T_1 T_2 \Gamma(\alpha) \sum_{n=0}^\infty C_n f_n(x) t^n (\alpha)_{\mu n} H_{A+1, C+1:[P_1, Q_1];\ldots;[P_r, Q_r]}^{0, \lambda+1:(M_1, N_1);\ldots;(M_r, N_r)}$$

$$\cdot \left[\begin{array}{llll} [(1-\beta-\eta n):\xi_1,\ldots,\xi_r,1,\ldots,1,1,\ldots,1] & :[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; \\ [(1-\beta-\alpha-\eta n-\mu n):\xi_1,\ldots,\xi_r,1,\ldots,1,1,\ldots,1] : [b'_j;B'_j];\ldots;[b^r_j;B^r_j]; \end{array} Z_1^{\xi_1},\ldots,Z_r^{\xi_r} \right], \quad (21)$$

where

$$T_1 = \prod_{j=1}^{M} \left[1 - \frac{2k_j^2}{k_j^2 - 1} \right]^{-L_j} \tag{21a}$$

$$T_2 = \prod_{i=1}^{N} \left[1 - \frac{(\delta_i - \rho_i)}{(1+\delta_i)} \right]^{-\lambda_j} \tag{21b}$$

provided that

$$\min\{\mathrm{Re}\,(\alpha),\mathrm{Re}\,(\beta)\} > 0, \ \mathrm{Re}\,(\eta) > 0, \ \mathrm{Re}\,(\mu) > 0, \ \delta_i,\rho_i,\lambda_i,L_j \in C; \ |k_j| < 1, \ (i=1,\ldots,N, \ j=1,\ldots,M),$$

$$\max\left\{ |\rho_i|,|\delta_i|,\left|\frac{2k_j^2}{k_j^2-1}\right|,\left|\frac{\delta_i-\rho_i}{1+\delta_i}\right| \right\} < 1, \ \max|z_\ell| \to 0, \ (\ell=1,\ldots,r) \ \text{and} \ \min|z_\ell| \to \infty(\ell=1,\ldots,r).$$

On taking $\xi_\ell = 1(\ell = 1,\ldots,r)$, we obtain the following corollary:

Corollary 3. *Let the generating function $F(x,t)$ and $H_{P_\ell,Q_\ell}^{M_\ell,N_\ell}(z)$ be the H-function of one variable defined in* (12) *and* (13) *respectively. Then*

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{\ell=1}^{r} H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell) \left| \begin{array}{l} (a_j^\ell,A_j^\ell)_{1,P_\ell} \\ (b_j^\ell,B_j^\ell)_{1,Q_\ell} \end{array} \right. \right]$$

$$\cdot \prod_{j=1}^{M} \left[1 - \frac{2\omega k_j^2}{k_j^2-1} \right]^{-L_j} \prod_{i=1}^{N} \left[1 - \frac{(\delta_i-\rho_i)\omega}{(1+\delta_i)} \right]^{-\lambda_i} F[x,t\omega^\eta(1-\omega)^\mu]d\omega$$

$$= T_1 T_2 \Gamma(\alpha) \sum_{n=0}^{\infty} C_n f_n(x) t^n (\alpha)_{\mu n} H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}$$

$$\cdot \left[\begin{array}{l} [(1-\beta-\eta n):[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; \\ [(1-\beta-\alpha-\eta n-\mu n):[b'_j;B'_j];\ldots;[b^r_j;B^r_j]; \end{array} Z_1,\ldots,Z_r \right], \quad (22)$$

valid under the condition as needed for (21).

Particularly, on taking $\xi_1 = \cdots = \xi_r = A'_j = \cdots = A^r_j = B'_j = \cdots = B^r_j = \eta = \mu = 1; \ \delta_i = \rho_i;$ $\lambda_i = 0$ for $(1,\ldots,N-1)$, $\rho_N = O; \delta_N = \delta, \ \lambda_n = \gamma$, following corollary can be stated:

Corollary 4. *Let the generating function $F(x,t)$ and $H_{P_\ell,Q_\ell}^{M_\ell,N_\ell}(z)$ be the H-function of one variable defined in* (12) *and* (13) *respectively. Then*

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{\ell=1}^{r} H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell) \left| \begin{array}{l} (a_j^\ell,A_j^\ell)_{1,P_\ell} \\ (b_j^\ell,B_j^\ell)_{1,Q_\ell} \end{array} \right. \right]$$

$$\cdot \prod_{j=1}^{M} \left[1 - \frac{2\omega k_j^2}{k_j^2 - 1}\right]^{-L_j} \left[1 - \frac{\delta\omega}{1+\delta}\right]^{-\gamma} F[x, t\omega(1-\omega)]\, d\omega$$

$$= W_1 W_2 \Gamma(\alpha) \sum_{n=0}^{\infty} C_n f_n(x) t^n (\alpha)_n G_{A+1,C+1:[P_1,Q_1];...;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);...;(M_r,N_r)}$$

$$\cdot \left[\begin{array}{c} [(1-\beta-n):a_j';a_j^r; \\ [(1-\beta-\alpha-2n):b_j';b_j^r; \end{array} Z_1,...,Z_r \right], \tag{23}$$

where

$$W_1 = \prod_{j=1}^{M} \left[1 - \frac{2k_j^2}{k_j^2 - 1}\right]^{-L_j} \tag{24a}$$

$$W_2 = \left[1 - \frac{\delta}{1+\delta}\right]^{-\gamma} \tag{24b}$$

provided that

$$\text{Re}(\alpha),\ \text{Re}(\beta) > O,\ L_j, \gamma \in C;\ |k_j| < 1,\ \max\left\{\left|\frac{2k_j^2}{k_j^2 - 1}\right|, \left|\frac{\delta}{1+\delta}\right|\right\} < 1,\ (j=1,...,M\ i=1,...,N).$$

Proofs. Expressing $F(x, t)$ by its power series from (12) in the integral (18), changing the order of integration and summation, which is permissible ,due to uniform convergence of the series involved. Using the definition (14) of the H-function of several variables in the evaluation of the resulting integral, we get the result (18), which proves Theorem 1.   □

The proofs of Theorem 2 and Corollaries 1, 2, 3 and 4 are similar to that of the Theorem 1.

3. Applications

In view of the importance and usefulness of the theorems and corollaries discussed in the last section, we mention some interesting applications, which indicates manifold generality of the results obtained in this article.

(I) Consider the generating function [34]

$$F(x, t) = (1-x)^{-\sigma} \sum_{n=0}^{\infty} (\sigma)_n \frac{x^n t^n}{n!}, \tag{25}$$

and by the use of the Theorem 1, under the conditions stated in (18), we get the following interesting results:

$$\int_0^1 \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^{r} H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell)^{\xi_\ell} \left| \begin{array}{c} (a_j^\ell, A_j^\ell)_{1,P_\ell} \\ (b_j^\ell, B_j^\ell)_{1,Q_\ell} \end{array} \right. \right] [1 - xt\omega^\eta(1-\omega)^\mu]^{-\sigma}\, d\omega$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^{\infty} (\gamma - \alpha)_{\mu n} (\sigma)_n \frac{x^n t^n}{n!}$$

$$\cdot H^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]} \left[\begin{array}{c} [(1-\alpha-\eta n):\xi_1,\ldots,\xi_r]:[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; \\ [(1-\gamma-\eta n-\mu n):\xi_1,\ldots,\xi_r]:[b'_j;B'_j];\ldots;[b^r_j;B^r_j]; \end{array} \; Z_1^{\xi_1},\ldots,Z_r^{\xi_r} \right], \quad (26)$$

which on setting $\omega = \cos^2(\theta/2)$ and $\cos\theta = 2\cos^2(\theta/2) - 1$, the above equation (26) gives the following generalization of the elliptic-type integral

$$\int_0^1 \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right) \prod_{\ell=1}^r H^{M_\ell,N_\ell}_{P_\ell,Q_\ell} \left[\left(\cos^2\left(\frac{\theta}{2}\right) z_\ell\right)^{\xi_\ell} \Bigg| \begin{array}{c} (a^\ell_j, A^\ell_j)_{1,P_\ell} \\ (b^\ell_j, B^\ell_j)_{1,Q_\ell} \end{array} \right]$$

$$\cdot \left[1 - xt\cos^{2\eta}\left(\frac{\theta}{2}\right) \sin^{2\mu}\left(\frac{\theta}{2}\right) \right]^{-\sigma} d\theta$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^{\infty} (\gamma - \alpha)_{\mu n} (\sigma)_n \frac{x^n t^n}{n!}$$

$$\cdot H^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]} \left[\begin{array}{c} [(1-\alpha-\eta n):\xi_1,\ldots,\xi_r]:[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; \\ [(1-\gamma-\eta n-\mu n):\xi_1,\ldots,\xi_r]:[b'_j;B'_j];\ldots;[b^r_j;B^r_j]; \end{array} \; Z_1^{\xi_1},\ldots,Z_r^{\xi_r} \right]. (27)$$

On setting $\omega = \sin^2(\theta/2)$ and using $\cos\theta = 1 - 2\sin^2(\theta/2)$, $\sigma \to 0$ in (26), we get the following generalized family of elliptic-type integrals:

$$\int_0^1 \sin^{2\alpha-1}\left(\frac{\theta}{2}\right) \cos^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right) \prod_{\ell=1}^r H^{M_\ell,N_\ell}_{P_\ell,Q_\ell} \left[\left(\sin^2\left(\frac{\theta}{2}\right) z_\ell\right)^{\xi_\ell} \Bigg| \begin{array}{c} (a^\ell_j, A^\ell_j)_{1,P_\ell} \\ (b^\ell_j, B^\ell_j)_{1,Q_\ell} \end{array} \right] d\theta$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^{\infty} (\gamma - \alpha)_{\mu n} (\sigma)_n \frac{x^n t^n}{n!}$$

$$\cdot H^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]} \left[\begin{array}{c} [(1-\alpha):\xi_1,\ldots,\xi_r]:[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; \\ [(1-\gamma):\xi_1,\ldots,\xi_r]:[b'_j;B'_j];\ldots;[b^r_j;B^r_j]; \end{array} \; Z_1^{\xi_1},\ldots,Z_r^{\xi_r} \right]. \quad (28)$$

It can be seen that the above elliptic-type integral (26) also provides generalization to a number of known elliptic-type integrals and the well known complete elliptic-type integrals [3] of the first kind.

Also, by the application of the Theorem 2, under the conditions stated for (21) and generating function (25), we have obtained the following new family of elliptio-type integrals, which also generalizes a number of known families of elliptic - type integrals

$$\int_0^1 \omega^{\beta-1} (1-\omega)^{\alpha-1} \prod_{\ell=1}^r H^{M_\ell,N_\ell}_{P_\ell,Q_\ell} \left[(\omega z_\ell)^{z_\ell} \Bigg| \begin{array}{c} (a^\ell_j, A^\ell_j)_{1,P_\ell} \\ (b^\ell_j, B^\ell_j)_{1,Q_\ell} \end{array} \right]$$

$$\cdot \prod_{j=1}^M \left[1 - \frac{2\omega k_j^2}{k_j^2-1} \right]^{-L_j} \prod_{i=1}^N \left[1 - \frac{(\delta_i - \rho_i)\omega}{(1+\delta_i)} \right]^{-\lambda_i} [1 - xt\omega^\eta(1-\omega)^\mu]^{-\sigma} d\omega$$

$$= T_1 T_2 \Gamma(\alpha) \sum_{n=0}^{\infty} (\alpha)_{\mu n} (\sigma)_n \frac{x^n t^n}{n!} H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}$$

$$\cdot \begin{bmatrix} [(1-\beta-\eta n):\xi_1,\ldots,\xi_r,1,\ldots,1,1,\ldots,1]:[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; & \\ [(1-\beta-\alpha-\eta n-\mu n):\xi_1,\ldots,\xi_r,1,\ldots,1,1,\ldots,1]:[b'_j;B'_j];\ldots;[b^r_j;B^r_j]; & (Z_1)^{\xi_1},\ldots,(Z_r)^{\xi_r} \end{bmatrix}. (29)$$

For $\sigma \to 0$ and $\omega = \sin^2(\frac{\theta}{2})$ with $\xi_1 = \cdots = \xi_r = A'_j = \cdots = A^r_j = B'_j = \cdots = B^r_j = 1$, above (29) yields the following explicit representation of generalized family of elliptic-type integral

$$\int_0^1 \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) \prod_{\ell=1}^r H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[\left(\sin^2\left(\frac{\theta}{2}\right) z_\ell\right) \Bigg| \begin{matrix} (a^\ell_j;1)_{1,P_\ell} \\ (b^\ell_j;1)_{1,Q_\ell} \end{matrix} \right]$$

$$\cdot \prod_{j=1}^M \left[1 - \frac{2\sin^2\left(\frac{\theta}{2}\right)k_j^2}{k_j^2-1}\right]^{-L_j} \prod_{i=1}^N \left[1 - \frac{(\delta_i-\rho_i)\sin^2\left(\frac{\theta}{2}\right)}{(1+\delta_i)}\right]^{-\lambda_i}$$

$$= T_1 T_2 G_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)} \left[\begin{matrix} [(1-\beta):a'_j;\ldots;a^r_j; \\ [(1-\alpha-\beta):b'_j;\ldots;b^r_j; \end{matrix} Z_1,\ldots,Z_r \right]. \quad (30)$$

also for $\sigma \to 0$ and $\omega = \sin^2(\frac{\theta}{2})$ with $\xi_1 = \cdots = \xi_r = A'_j = \cdots = A^r_j = B'_j = \cdots = B^r_j = 1$; $(z_\ell)^{\xi_\ell} = y_\ell$; for $\ell = 1,2,\ldots,r$; $\delta_i = \rho_i = 0$ or $i = l,2,\ldots,(N-1)$ and $\rho_N = 0$; $\delta_N = \delta$; $\lambda_i = \gamma$, above (29) yields the following explicit representation of generalized family of elliptic-type integral

$$\int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) \prod_{\ell=1}^r H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[\left(\sin^2\left(\frac{\theta}{2}\right) y_\ell\right) \Bigg| \begin{matrix} (a^\ell_j;1)_{1,P_\ell} \\ (b^\ell_j;1)_{1,Q_\ell} \end{matrix} \right]$$

$$\cdot \prod_{j=1}^M \left[1 - \frac{2k_j^2\sin^2\left(\frac{\theta}{2}\right)}{k_j^2-1}\right]^{-L_j} \prod_{i=1}^N \left[1 - \frac{\delta(\sin^2\frac{\theta}{2})}{1+\delta}\right]^{-\gamma} d\theta$$

$$= \Gamma(\alpha) W_1 W_2 G_{A+1,C+1}^{0,\lambda+1} \left[\begin{matrix} [(1-\beta-n):a'_j;\ldots;a^r_j; \\ [(1-\alpha-\beta-n):b'_j;\ldots;b^r_j; \end{matrix} y_1,\ldots,y_r \right]. \quad (31)$$

(II) Consider the generating relation [34]

$$F(x,t) = (1-X_1 t)^{-\alpha_1}(1-X_2 t)^{-\alpha_2} = \sum_{n=0}^{\infty} g_n^{\alpha_1,\alpha_2}(X_1,X_2) t^n \quad (32)$$

where $g_n^{\alpha_1,\alpha_2}$ the Lagrange polynomial defined by

$$g_n^{\alpha_1,\alpha_2}(x,y) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_{n-r}}{r!(n-r)!} x^r y^{n-r}, \quad (33)$$

and by the application of Theorem 1, under the conditions stated in (18), we get

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^r H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell)^{z_\ell} \Bigg| \begin{matrix} (a^\ell_j,A^\ell_j)_{1,P_\ell} \\ (b^\ell_j,B^\ell_j)_{1,Q_\ell} \end{matrix} \right] \prod_{j=1}^2 [1 - X_j t\omega^\eta(1-\omega)^\mu]^{-\sigma_j} d\omega$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^{\infty} (\gamma - \alpha)_{\mu n} g_n^{\sigma_1,\sigma_2}(X_1, X_2) t^n$$

$$\cdot H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)} \left[\begin{array}{c} [(1-\alpha-\eta n):\xi_1,\ldots,\xi_r]:[a_j';A_j'];\ldots;[a_j^r;A_j^r]; \\ [(1-\gamma-\eta n-\mu n):\xi_1,\ldots,\xi_r]:[b_j';B_j'];\ldots;[b_j^r;B_j^r]; \end{array} Z_1^{\xi_1},\ldots,Z_r^{\xi_r} \right], \quad (34)$$

Also, by the application of the Theorem 2 under the conditions stated in (21) and making use of the generating relation (32),we obtain

$$\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{\ell=1}^{r} H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell)^{z_\ell} \left| \begin{array}{c} (a_j^\ell,A_j^\ell)_{1,P_\ell} \\ (b_j^\ell,B_j^\ell)_{1,Q_\ell} \end{array} \right. \right]$$

$$\cdot \prod_{j=1}^{M} \left[1 - \frac{2\omega k_j^2}{k_j^2 - 1} \right]^{-L_j} \prod_{i=1}^{N} \left[1 - \frac{(\delta_i - \rho_i)\omega}{(1+\delta_i)} \right]^{-\lambda_i} \prod_{q=1}^{2} [1 - X_q t \omega^\eta (1-\omega)^\mu]^{-\sigma_q} d\omega$$

$$= T_1 T_2 \Gamma(\alpha) \sum_{n=0}^{\infty} g_n^{\sigma_1,\sigma_2}(X_1, X_2) t^n (\alpha)_{\mu n} H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}$$

$$\cdot \left[\begin{array}{c} [(1-\beta-\eta n):\xi_1,\ldots,\xi_r,1,\ldots,1,1,\ldots,1]:[a_j';A_j'];\ldots;[a_j^r;A_j^r]; \\ [(1-\beta-\alpha-\eta n-\mu n):\xi_1,\ldots,\xi_r,1,\ldots,1,1,\ldots,1]:[b_j';B_j'];\ldots;[b_j^r;B_j^r]; \end{array} (Z_1)^{\xi_1},\ldots,(Z_r)^{\xi_r} \right]. \quad (35)$$

(III) Consider the well-known generating function

$$F(x, t) = e^{-xt} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{n!}, \quad (36)$$

and by the use of the Theorem 1, under the conditions stated with (18), we find

$$\int_0^1 e^{-xt[\omega^\eta(1-\omega)^\mu]} \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^{r} H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell)^{\xi_\ell} \left| \begin{array}{c} (a_j^\ell,A_j^\ell)_{1,P_\ell} \\ (b_j^\ell,B_j^\ell)_{1,Q_\ell} \end{array} \right. \right] d\omega$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^{\infty} (\gamma - \alpha)_{\mu n} \frac{(-1)^n x^n t^n}{n!} H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}$$

$$\cdot \left[\begin{array}{c} [(1-\sigma-\eta n):\xi_1,\ldots,\xi_r]:[a_j';A_j'];\ldots;[a_j^r;A_j^r]; \\ [(1-\gamma-\eta n-\mu n):\xi_1,\ldots,\xi_r]:[b_j';B_j'];\ldots;[b_j^r;B_j^r]; \end{array} (Z_1)^{\xi_1},\ldots,(Z_r)^{\xi_r} \right], \quad (37)$$

Also, by the application of Theorem 2 with the condition surrounding (21) and on using (36), we have the following useful integral

$$\int_0^1 e^{-xt}[\omega^\eta(1-\omega)^\mu] \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{\ell=1}^{r} H_{P_\ell,Q_\ell}^{M_\ell,N_\ell} \left[(\omega z_\ell)^{\xi_\ell} \left| \begin{array}{c} (a_j^\ell,A_j^\ell)_{1,P_\ell} \\ (b_j^\ell,B_j^\ell)_{1,Q_\ell} \end{array} \right. \right]$$

$$\cdot \prod_{j=1}^{M} \left[1 - \frac{2\omega k_j^2}{k_j^2 - 1} \right]^{-L_j} \prod_{i=1}^{N} \left[1 - \frac{(\delta_i - \rho_i)\omega}{(1+\delta_i)} \right]^{-\lambda_i} d\omega$$

$$= T_1 T_2 \Gamma(\alpha) \sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n (\alpha)_{\mu n}}{n!} H_{A+1,C+1:[P_1,Q_1];\ldots;[P_r,Q_r]}^{0,\lambda+1:(M_1,N_1);\ldots;(M_r,N_r)}$$

$$\cdot \begin{bmatrix} [(1-\beta-\eta n):\xi_1,\ldots,\xi_r]:[a'_j;A'_j];\ldots;[a^r_j;A^r_j]; \\ [(1-\beta-\alpha-\eta n-\mu n):\xi_1,\ldots,\xi_r]:[b'_j;B'_j];\ldots;[b^r_j,B^r_j]; \end{bmatrix} (Z_1)^{\xi_1},\ldots,(Z_r)^{\xi_r} \Big]. \qquad (38)$$

It can be seen that the theorems (Theorem 1 and Theorem 2 along with and their corollaries) discussed in this article provide generalizations to a number of the known elliptic-type integrals. Furthermore, these Theorems also have their wide applications in formulation of various new elliptic-type integrals and a great use in solving a very spacious class of Euler-type integrals in terms of different generalized functions.

Acknowledgements

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