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IMPROVEMENTS OF SOME INTEGRAL INEQUALITIES OF GRÜSS TYPE

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S. S. Dragomir ([1]) has proved the following results:

Theorem A. *Let $f,\ g : [a,b] \to \boldsymbol{R}$ be two differentiable mappings on (a,b). If $f' \in L_\alpha(a,b)$ and $g' \in L_\beta(a,b)$ with $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we have the inequality*

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y| \left|\int_x^y |f'(t)|^\alpha dt\right| dxdy\right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y| \left|\int_x^y |g'(t)|^\beta dt\right| dxdy\right)^{\frac{1}{\beta}} \\ &\leq \frac{1}{6}\|f'\|_\alpha \|g'\|_\beta (b-a). \end{aligned} \tag{1}$$

where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx.$$

The first inequality in (1) *is sharp.*

Theorem B. *Let $f,\ g : [a,b] \to \boldsymbol{R}$ be two differentiable mappings on (a,b). If $f' \in L_\infty(a,b)$ and $g' \in L_1(a,b)$ then we have the inequality*

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-y| \sup_{t\in[x,y]} |f'(t)| \left|\int_x^y |g'(z)|dz\right| dxdy \\ &\leq \frac{1}{6}\|f'\|_\infty \|g'\|_1 (b-a). \end{aligned} \tag{2}$$

The first inequality in (2) *is sharp.*

In this paper we shall improve the second inequalities in (1) and (2), i.e. the following theorem is valid:

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Theorem. (i) *Let the assumptions of Theorem A be fulfiled. Then*

$$|T(f,g)| \le \frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dxdy\right)^{\frac{1}{\alpha}}$$
$$\times\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y|\left|\int_x^y |g'(t)|^\beta dt\right| dxdy\right)^{\frac{1}{\beta}}$$
$$\le \frac{1}{8}\|f'\|_\alpha\|g'\|_\beta(b-a). \tag{1'}$$

(ii) *Let the assumptions os Theorem B be fulfilled. Then*

$$|T(f,g)| \le \frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-y| \sup_{t\in[x,y]}|f'(t)| \left|\int_x^y |g'(t)|dt\right| dxdy$$
$$\le \frac{1}{8}\|f'\|_\infty\|g'\|_1(b-a).$$

Proof. (i) Let us consider the integral

$$\int_a^b\int_a^b |x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dxdy$$
$$= 2\int_a^b\int_a^x (x-y)\int_x^y |f'(t)|^\alpha dtdydx$$
$$= 2\int_a^b\int_a^x\int_a^t (x-y)|f'(t)|^\alpha dydtdx$$
$$= \int_a^b\int_a^x \left[(x-a)^2-(x-t)^2\right]|f'(t)|^\alpha dtdx$$
$$= \int_a^b |f'(t)|^\alpha \int_t^b [(x-a)^2-(x-t)^2]dxdt$$
$$= \frac{1}{3}\int_a^b \left[(b-a)^3-(t-a)^3-(b-t)^3\right]|f'(t)|^\alpha dt$$

By elementary calculus we can obtain

$$(b-a)^3-(t-a)^3-(b-t)^3 \le \frac{3}{4}(b-a)^3.$$

Therefore, we have

$$\int_a^b\int_a^b |x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dxdy \le \frac{1}{4}(b-a)^3\int_a^b |f'(t)|^\alpha dt = \frac{1}{4}(b-a)^3\|f'\|_\alpha^\alpha.$$

It is clear that we also have

$$\int_a^b\int_a^b |x-y|\left|\int_x^y |g'(t)|^\beta dt\right| dxdy \le \frac{1}{4}(b-a)^3\int_a^b |g'(t)|^\beta dt = \frac{1}{4}(b-a)^3\|g'\|_\beta^\beta.$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dxdy\right)^{\frac{1}{\alpha}}\\
&\qquad\times\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y|\left|\int_x^y |g'(t)|^\beta dt\right| dxdy\right)^{\frac{1}{\beta}}\\
&\qquad\le \frac{1}{2}\left(\frac{1}{4}(b-a)\|f'\|_\alpha^\alpha\right)^{\frac{1}{\alpha}}\left(\frac{1}{4}(b-a)\|g'\|_\beta^\beta\right)^{\frac{1}{\beta}}\\
&\qquad= \frac{1}{8}\|f'\|_\alpha\|g'\|_\beta(b-a).
\end{aligned}$$

(ii) Similarly, we can obtain

$$\begin{aligned}
&\int_a^b\int_a^b |x-y| \sup_{t\in[x,y]}|f'(t)|\left|\int_x^y |g'(t)|dt\right| dxdy\\
&\qquad\le \|f'\|_\infty \int_a^b\int_a^b |x-y|\left|\int_x^y |g'(t)|dt\right| dxdy\\
&\qquad\le \frac{1}{4}\|f'\|_\infty\|g'\|_1(b-a)^3,
\end{aligned}$$

i.e. we have improvement of the second inequality in (2), that is

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |x-y| \sup_{t\in[x,y]}|f'(t)|\left|\int_x^y |g'(t)|dt\right| dxdy \le \frac{1}{8}\|f'\|_\infty\|g'\|_1(b-a).$$

Remark. It was prove in [2] (see also [3, p.202]):

$$|T(f,g)| \le \frac{b-a}{4}\left[\frac{2^\alpha-1}{\alpha(\alpha+1)}\right]^{\frac{1}{\alpha}}\left[\frac{2^\beta-1}{\beta(\beta+1)}\right]^{\frac{1}{\beta}}\|f'\|_\alpha\|g'\|_\beta$$

if $f'\in L_\alpha$, $g'\in L_\beta$, $\alpha\ge 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

For $\alpha=\beta=2$ we have

$$|T(f,g)| \le \frac{b-a}{8}\|f'\|_2\|g'\|_2, \tag{3}$$

while for $\alpha=1$, $\beta=\infty$ we get

$$|T(f,g)| \le \frac{b-a}{4}\|f'\|_1\|g'\|_\infty.$$

Results obtained in this paper show that constant $\frac{1}{8}$ is valid for all α, $1\le\alpha\le\infty$. On the other hand side it is well known that inequality (3) can be improved. Namely, A. Lupaş [4] has proved that

$$|T(f,g)| \le \frac{b-a}{\pi^2}\|f'\|_2\|f'\|_2.$$

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