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# PRESERVING PROPERTIES OF SUBORDINATION AND SUPERORDINATION OF ANALYTIC FUNCTIONS ASSOCIATED WITH A FRACTIONAL DIFFERINTEGRAL OPERATOR

JAMAL M. SHENAN

**Abstract**. In this paper, we obtain some subordination and superordination-preserving results of analytic functions associated with the fractional differintegral operator $U_{0,z}^{\alpha,\beta,\gamma}$. Sandwich-type result involving this operator is also derived.

## 1. Introduction

Let $H(U)$ be the class of functions analytic in $U = \{z : z \in C \text{ and } |z| < 1\}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \cdots$, with $H_0 \equiv H[0,1]$ and $H \equiv H[1,1]$.

Let $A_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \left(p, \in \mathbb{N} = \{1,2,3,\ldots\}; z \in U\right), \tag{1.1}$$

which are analytic in the open unit disk $U$.

Let $f$ and $F$ be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if $F$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$ (see [5, 6]).

Let $\Psi : C^2 \times U \to C$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the first order differential subordination

$$\Psi\left(p(z), zp'(z); z\right) \prec h(z) \, (z \in U), \tag{1.2}$$

then $p$ is called a solution of the differential subordination (1.2).

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The univalent function $q$ is called a dominant solutions of the differential subordination (1.2) if $p \prec q$ for all $p$ satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1.2) is said to be the best dominant of (1.2).

Similarly, let $\Phi : C^2 \times U \to C$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the first order differential superordination

$$h(z) \prec \Phi\left(p(z), zp'(z); z\right)(z \in U), \tag{1.3}$$

then $p$ is called a solution of the differential superordination (1.3).

The univalent function $q$ is called a subordinant solutions of the differential superordination (1.3) if $q \prec p$ for all $p$ satisfying (1.3). A subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinant $q$ of (1.3) is said to be the best subordinant. ( see the monograph by Miller and Mocanu [7], and [8]).

We recall the definitions of the fractional derivative and integral operators introduced and studied by Saigo (cf. [14], [15]).

**Definition 1.** Let $\alpha > 0$ and $\beta, \gamma \in R$, then the generalized fractional integral operator $\mathrm{I}_{0,z}^{\alpha,\beta,\gamma}$ of order $\alpha$ of a function $f(z)$ is defined by

$$\mathrm{I}_{0,z}^{\alpha,\beta,\gamma} f(z) = \frac{z^{-\alpha-\beta,}}{\Gamma(\alpha)} \int_0^z (z-t)_2^{\alpha-1} F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{t}{z}\right) f(t)\,dt, \tag{1.4}$$

where the function $f(z)$ is analytic in a simply-connected region of the $z$- plane containing the origin and the multiplicity of $(z-t)^{(\alpha-1)}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$ provided further that

$$f(z) = O(|z|^{\varepsilon}), z \to 0 \quad \text{for } \varepsilon > \max\left(0, \beta-\gamma\right) - 1. \tag{1.5}$$

**Definition 2.** Let $0 \leq \alpha < 1$ and $\beta, \gamma \in R$, then the generalized fractional derivative operator $J_{0,z}^{\alpha,\beta,\gamma}$ of order $\alpha$ of a function $f(z)$ defined by

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\gamma} f(z) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\left[z^{\alpha-\beta} \int_0^z (z-t)_2^{-\alpha} F_1\left(\beta-\alpha, 1-\gamma; 1-\alpha; 1-\frac{t}{z}\right) f(t)\,dt\right], \\ &= \frac{d^n}{dz^n} J_{0,z}^{\alpha-n,\beta,\gamma} f(z) \qquad (n \leq \alpha < n+1; n \in N), \end{aligned} \tag{1.6}$$

where the function $f(z)$ is analytic in a simply-connected region of the $z$- plane containing the origin, with the order as given in (1.5) and multiplicity of $(z-t)^{\alpha}$ is removed by requiring $\log(z-t)$ to be real when$(z-t) > 0$.

Note that

$$I_{0,z}^{\alpha,-\alpha,\gamma} f(z) = D_z^{-\alpha} f(z)(\alpha > 0), \tag{1.7}$$

and

$$J_{0,z}^{\alpha,\alpha,\gamma} f(z) = D_z^{\alpha} f(z) (0 \leq \alpha < 1), \tag{1.8}$$

where $D_z^{-\alpha} f(z)$ and $D_z^{\alpha} f(z)$ are respectively the well known Riemann-Liouvill fractional integral and derivative operators (cf. [10] and [11], see also [16]).

**Definition 3.** For real number $\alpha\,(-\infty < \alpha < 1)$ and $\beta\,(-\infty < \beta < 1)$ and a positive real number $\gamma$, the fractional operator $U_{0,z}^{\alpha,\beta,\gamma}\,\_ : A_p \to A_p$ is defined in terms of $J_{0,z}^{\alpha,\beta,\gamma}$ and $\mathrm{I}_{0,z}^{\alpha,\beta,\gamma}$ by (see [9] and [4])

$$U_{0,z}^{\alpha,\beta,\gamma} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p}, \tag{1.9}$$

which for $f(z) \neq 0$may be written as

$$U_{0,z}^{\alpha,\beta,\gamma} f(z) = \begin{cases} \dfrac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} J_{0,z}^{\alpha,\beta,\gamma} f(z); & 0 \leq \alpha \leq 1 \\ \dfrac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} \mathrm{I}_{0,z}^{-\alpha,\beta,\gamma} f(z); & -\infty \leq \alpha < 0 \end{cases} \tag{1.10}$$

where $J_{0,z}^{\alpha,\beta,\gamma} f(z)$and $\mathrm{I}_{0,z}^{-\alpha,\beta,\gamma} f(z)$ are, respectively the fractional derivative of $f$ of order $\alpha$ if $0 \leq \alpha < 1$ and the fractional integral of $f$ of order $-\alpha$ if $-\infty \leq \alpha < 0$.

It is easily verified (see Choi [3]) from (1.9) that

$$(p-\beta)\, U_{0,z}^{\alpha+1,\beta+1,\gamma+1} f(z) + \beta U_{0,z}^{\alpha,\beta,\gamma} f(z) = z \left( U_{0,z}^{\alpha,\beta,\gamma} f(z) \right)'. \tag{1.11}$$

Note that

$$U_{0,z}^{\alpha,\alpha,\gamma} f(z) = \Omega_z^{(\alpha,p)} f(z)\, (-\infty < \alpha < 1), \tag{1.12}$$

The fractional differintegral operator $\Omega_z^{(\alpha,p)} f(z)$ for $(-\infty < \alpha < p+1)$ is studied by Patel and Mishra [12], and the fractional differential operator $\Omega_z^{(\alpha,p)}$ with $0 \leq \alpha < 1$ was investigated by Srivastava and Aouf [17]. We, further observe that $\Omega_z^{(\alpha,1)} = \Omega_z^{\alpha}$ is the operator introduced and studied by Owa and Srivastava [11].

It is interesting to observe that

$$U_{0,z}^{0,0,\gamma} f(z) = f(z) \tag{1.13}$$

and

$$U_{0,z}^{1,1,\gamma} f(z) = \frac{z}{p} f'(z). \tag{1.14}$$

To prove our results, we need the following definitions and lemmas.

**Definition 4** ([7]). Denote by $Q$ the set of all functions $q(z)$ that are analytic and injective on $\bar{U}/E(q)$ where

$$E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U/E(q)$. Further let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

**Definition 5** ([8]). A function $L(z,t)\,(z \in U,\ t \geq 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in $z \in U$for all $t \geq 0$, $L(z,0)$ is continuously differentiable on $[0;1]$ for all $z \in U$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

**Lemma 1** ([13]). *The function $L(z,t) : U \times [0;1] \to \mathbb{C}$ of the form*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \qquad (a_1(t) \neq 0;\ t \geq 0),$$

*and $\lim_{t\to\infty}|a_1(t)| = \infty$ is a subordination chain if and only if*

$$Re\left\{\frac{z\partial L(z,t)/\partial t}{\partial L(z,t)/\partial t}\right\} > 0 (z \in U,\ t \geq 0).$$

**Lemma 2** ([5]). *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition*

$$Re\{H(is;\, t)\} \leq 0$$

*for all real $s$ and for all $t \leq -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + a_n z^n + a_{n+1}z^{n+1} + \cdots$, is analytic in $U$ and $Re\{H(p(z);\, zp'(z))\} > 0\ \ (z \in U)$. then $Re\{p(z)\} > 0$ for $z \in U$.*

**Lemma 3** ([6]). *Let $k, \gamma \in \mathbb{C}$ with $k \neq 0$ and let $h \in H(U)$ with $H(0) = c$. If $Re\{kh(z) + \gamma\} > 0\ \ (z \in U)$, then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{kq(z)+\gamma} = h(z) (z \in U;\ q(0) = c),$$

*is analytic in $U$ and satisfies $Re\{kh(z) + \gamma\} > 0$ for $z \in U$.*

**Lemma 4** ([7]). *Let $p \in Q(a)$ and let $q(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$, be analytic in $U$ with $q(z) \neq 0$ and $n \geq 1$. If $q$ is not subordinate to $p$, the there exists two points $z_0 = r_0 e^{i\theta} \in U$ and $\xi_0 \in \partial U/E(q)$ such that $q(U_{r_o}) \subset p(U)$; $q(z_0) = p(\xi_0)$ and $z_0 p'(z_0) = m\xi_0 p(\xi_0)$ $m \geq n$.*

**Lemma 5** ([8]). *Let $q \in H[a,1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$ also $\phi\left(q(z), zq'(z)\right) = h(z)$. If $L(z,t) = \phi(q(z), tzq'(z))$ is a subordination chain and $q \in H[a,1] \bigcap Q(a)$, then*

$$h(z) \prec \phi\left(p(z), zp'(z)\right),$$

*implies that $q(z) \prec p(z)$. Further if $\phi\left(q(z), zq'(z)\right) = h(z)$ has a univalent solution $q \in Q(a)$, then $q$ is the best subordination.*

In the present paper, we aim to prove some subordination-preserving and superordination -preserving properties associated with the fractional differintegral operator $U_{0,z}^{\alpha,\beta,\gamma}$. Sandwich-type result involving this operator is also derived. A simililar problem for analytic functions was studied by Aouf and Seoudy [1] and [2].

## 2. Subordination, superordination and sandwich results involving the operator $U_{0,z}^{\alpha,\beta,\gamma}$

**Theorem 1.** *Let $f, g \in A_p$ and let*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{2.1}$$

*where*

$$\phi(z) = \left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} g(z)}{U_{0,z}^{\alpha,\beta,\gamma} g(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g(z)}{z^p}\right)^{\mu} \tag{2.2}$$

$$\left(-\infty < \alpha < 1; -\infty < \beta < 1; \gamma \in \mathbb{R}^+; \mu > 0; z \in U\right),$$

*and $\delta$ is given by*

$$\delta = \frac{1+\mu^2(p-\beta)^2 - \left|1-\mu^2(p-\beta)^2\right|}{4\mu(p-\beta)}. \tag{2.3}$$

*Then the subordination condition*

$$\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} f(z)}{U_{0,z}^{\alpha,\beta,\gamma} f(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu} \prec \left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} g(z)}{U_{0,z}^{\alpha,\beta,\gamma} g(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu} \prec \left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g(z)}{z^p}\right)^{\mu},$$

*and the function $\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g(z)}{z^p}\right)^{\mu}$ is the best dominant.*

**Proof.** Let us define the functions $F(z)$ and $G(z)$ in $U$ by

$$F(z) = \left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu} \quad \text{and} \quad G(z) = \left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g(z)}{z^p}\right)^{\mu} \ (z \in U), \tag{2.4}$$

we assume here, without loss of generality, that $G(z)$ is analytic and univalent on $\overline{U}$ and

$$G'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0<\rho<1$. These new functions have the desired properties on $\overline{U}$, and we can use them in the proof of our result. Therefore, the results would follow by letting$\rho \to 1$.We first show that, if

$$q(z)=1+\frac{zG''(z)}{G'(z)} \quad (z \in U), \tag{2.5}$$

then

$$\Re\left\{q(z)\right\}>0 \ \ (z\in U).$$

From (1.11) and the definition of the functions $G$, $\phi$, we obtain that

$$\phi(z)=G(z)+\frac{zG'(z)}{\mu(p-\beta)}. \tag{2.6}$$

Differentiating both side of (2.6) with respect to $z$ yields

$$\phi'(z)=\left(1+\frac{1}{\mu(p-\beta)}\right)G'(z)+\frac{zG'(z)}{\mu(p-\beta)}. \tag{2.7}$$

Combining (2.5) and (2.7), we easily get

$$1+\frac{z\phi''(z)}{\phi'(z)}=q(z)+\frac{zq'(z)}{q(z)+\mu(p-\beta)}=h(z) \quad (z\in U). \tag{2.8}$$

It follows from (2.1) and (2.8) that

$$Re\left\{h(z)+\mu(p-\beta)\right\}>0 \ \ (z\in U). \tag{2.9}$$

Moreover, by using Lemma 3, we conclude that the differential equation (2.8) has a solution $q(z)\in H(U)$ with $h(0)=q(0)=1$. Let

$$H(u,v)=u+\frac{v}{u+\mu(p-\beta)}+\delta,$$

where $\delta$ is given by (2.3). From (2.8) and (2.9), we obtain

$$Re\left\{H(q(z);zq'(z))\right\}>0 \ \ (z\in U).$$

To verify the condition that

$$Re\{H(is;t)\}\leq 0 \quad \left(t\leq -(1+s^2)/2;\ s\in\mathbb{R}\right). \tag{2.10}$$

we proceed it as follows:

$$Re\{H(is;t)\} = Re\left\{is+\frac{t}{is+\mu(p-\beta)}+\delta\right\} = \frac{t\mu(p-\beta)}{s^2+\mu^2(p-\beta)^2}+\delta$$

$$\leq -\frac{\psi_p(\beta,\mu,\delta,s)}{2\left[s^2+\mu^2(p-\beta)^2\right]},$$

where

$$\psi_p(\beta,\mu,\delta,s) = \left[\mu(p-\beta)-2\delta\right]s^2 - 2\delta\mu^2(p-\beta)^2 + \mu(p-\beta). \tag{2.11}$$

For $\delta$ given by (2.3), we note that the expression $\psi_p(\beta,\mu,\delta,s)$ in (2.11) is a positive, which implies that (2.10) holds. Thus, by using Lemma 2, we conclude that

$$Re\left\{q(z)\right\} > 0 \ \ (z \in U).$$

By the definition of $q(z)$, we know that $G$ is convex. To prove $F \prec G$, let the function $L(z,t)$ be defined by

$$L(z,t) = G(z) + \frac{(1+t)zG'(z)}{\mu(p-\beta)} \quad (0 \leq t < \infty;\, z \in U). \tag{2.12}$$

Since $G$ is convex, then

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1+\frac{(1+t)}{\mu(p-\beta)}\right) \neq 0 \quad (0 \leq t < \infty;\, z \in U)$$

and

$$Re\left\{\frac{z\partial L(z,t)/\partial t}{\partial L(z,t)/\partial t}\right\} = Re\left\{\mu(p-\beta)+(1+t)q(z)\right\} > 0 \ (0 \leq t < \infty;\, z \in U).$$

Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{zG'(z)}{\mu(p-\beta)} = L(z,0),$$

and

$$L(z,0) \prec L(z,t) \quad (0 \leq t < \infty),$$

which implies

$$L(\zeta,t) \notin L(U,0) \quad (0 \leq t < \infty; \zeta \in \partial U), \tag{2.13}$$

If $F$ is not subordinate to $G$, by using Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0)\, and\, z_0F'(z_0) = (1+t)\zeta_0 p(\zeta_0)\ (0 \leq t < \infty). \tag{2.14}$$

Hence, by virtue of (1.11) and (2.14), we have

$$\begin{aligned} L(\zeta_0,t) &= G(\zeta_0) + \frac{(1+t)zG'(\zeta_0)}{\mu(p-\beta)} = F(z_0) + \frac{z_0F'(z_0)}{\mu(p-\beta)} \\ &= \left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z_0)}{U_{0,z}^{\alpha,\beta,\gamma}f(z_0)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z_0)}{z_0^p}\right)^{\mu} \in \phi(U). \end{aligned}$$

This contradicts to (2.13). Thus, we deduce that$F \prec G$. Considering $F = G$, we see that the function $G$ is the best dominant. This completes the proof of Theorem 1.

By taking $\alpha = \beta$ in Theorem 1 and using the relation (1.12) we get the following Corollary

**Corollary 1.** *Let $f, g \in A_p$ and let*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{2.15}$$

*where*

$$\phi(z) = \left(\frac{\Omega_z^{(\alpha+1,p)} g(z)}{\Omega_z^{(\alpha,p)} g(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} g(z)}{z^p}\right)^{\mu} \left(-\infty < \alpha < 1; \mu > 0; z \in U\right), \tag{2.16}$$

*and $\delta$ is given by*

$$\delta = \frac{1 + \mu^2(p-\alpha)^2 - \left|1 - \mu^2(p-\alpha)^2\right|}{4\mu(p-\alpha)}. \tag{2.17}$$

*Then the subordination condition*

$$\left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha+1,p)} g(z)}{\Omega_z^{(\alpha,p)} g(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} g(z)}{z^p}\right)^{\mu},$$

*implies that* $\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha,p)} g(z)}{z^p}\right)^{\mu}$, *and the function* $\left(\frac{\Omega_z^{(\alpha,p)} g(z)}{z^p}\right)^{\mu}$ *is the best dominant.*

By taking $\alpha = 0$ in Corollary 1 and using the relation (1.13) and (1.14) we get the following Corollary

**Corollary 2.** *Let $f, g \in A_p$ and let*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{2.18}$$

*where*

$$\phi(z) = \left(\frac{zg'(z)}{pg(z)}\right)\left(\frac{g(z)}{z^p}\right)^{\mu} \left(\mu > 0; z \in U\right), \tag{2.19}$$

*and $\delta$ is given by*

$$\delta = \frac{1 + \mu^2 p^2 - \left|1 - \mu^2 p^2\right|}{4\mu p}. \tag{2.20}$$

*Then the subordination condition*

$$\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{f(z)}{z^p}\right)^{\mu} \prec \left(\frac{zg'(z)}{pg(z)}\right)\left(\frac{g(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{f(z)}{z^p}\right)^{\mu} \prec \left(\frac{g(z)}{z^p}\right)^{\mu},$$

*and the function* $\left(\frac{g(z)}{z^p}\right)^{\mu}$ *is the best dominant.*

We now derive the following superordination result.

**Theorem 2.** *Let $f, g \in A_p$ and let*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{2.21}$$

*where*

$$\phi(z)=\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}g(z)}{U_{0,z}^{\alpha,\beta,\gamma}g(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}g(z)}{z^p}\right)^{\mu} \tag{2.22}$$
$$\left(-\infty<\alpha<1;-\infty<\beta<1;\gamma\in\mathbb{R}^+;\mu>0;z\in U\right),$$

*and $\delta$ is given by (2.3). If the function $\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{U_{0,z}^{\alpha,\beta,\gamma}f(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right)^{\mu}$ is univalent in $U$ and $\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right)^{\mu}\in Q$, then the superordination condition*

$$\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}g(z)}{U_{0,z}^{\alpha,\beta,\gamma}g(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}g(z)}{z^p}\right)^{\mu}\prec\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f(z)}{U_{0,z}^{\alpha,\beta,\gamma}f(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}g(z)}{z^p}\right)^{\mu}\prec\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}f(z)}{z^p}\right)^{\mu},$$

*and the function $\left(\frac{U_{0,z}^{\alpha,\beta,\gamma}g(z)}{z^p}\right)^{\mu}$ is the best subordinan.*

**Proof.** Suppose that the functions $F$, $G$ and $q$ are defined by (2.4) and (2.5), respectively. By applying the similar method as in the proof of Theorem 1, we get

$$Re\{q(z)\}>0 \ \ (z\in U).$$

Next, to arrive at our desired result, we show that $G\prec F$. For this, we suppose that the function $L(z,t)$ be defined by (2.12).

Since $G$ is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G\prec F$. Moreover, since the differential equation

$$\phi(z)=G(z)+\frac{zG'(z)}{\mu(p-\beta)}=\varphi\left(G(z),\ zG'(z)\right)$$

has a univalent solution $G$, it is the best subordinant. This completes the proof.

By taking $\alpha=\beta$ in Theorem 2 and using the relation (1.12) we get the following Corollary

**Corollary 3.** *Let $f, g \in A_p$ and let*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{2.23}$$

*where*

$$\phi(z)=\left(\frac{\Omega_z^{(\alpha+1,p)}g(z)}{\Omega_z^{(\alpha,p)}g(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}g(z)}{z^p}\right)^{\mu}\left(-\infty<\alpha<1;\mu>0;z\in U\right), \tag{2.24}$$

*and $\delta$ is given by (2.3). If the function $\left(\frac{\Omega_z^{(\alpha+1,p)}g(z)}{\Omega_z^{(\alpha,p)}g(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}g(z)}{z^p}\right)^{\mu}$ is univalent in $U$ and $\left(\frac{\Omega_z^{(\alpha,p)}g(z)}{z^p}\right)^{\mu} \in Q$, then the superordination condition*

$$\left(\frac{\Omega_z^{(\alpha+1,p)}g(z)}{\Omega_z^{(\alpha,p)}g(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}g(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha+1,p)}f(z)}{\Omega_z^{(\alpha,p)}f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{\Omega_z^{(\alpha,p)}g(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha,p)}f(z)}{z^p}\right)^{\mu},$$

*and the function $\left(\frac{\Omega_z^{(\alpha,p)}g(z)}{z^p}\right)^{\mu}$ is the best subordinant.*

By taking $\alpha = 0$ in Corollary 3 and using the relation (1.13) and (1.14) we get the following corollary

**Corollary 4.** *Let $f, g \in A_p$ and let*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta, \tag{2.25}$$

*where*

$$\phi(z)=\left(\frac{zg'(z)}{pg(z)}\right)\left(\frac{g(z)}{z^p}\right)^{\mu}\left(\mu>0;z\in U\right), \tag{2.26}$$

*and $\delta$ is given by (2.3). If the function $\left(\frac{zg'(z)}{pg(z)}\right)\left(\frac{g(z)}{z^p}\right)^{\mu}$ is univalent in $U$ and $\left(\frac{g(z)}{z^p}\right)^{\mu} \in Q$, then the superordination condition*

$$\left(\frac{zg'(z)}{pg(z)}\right)\left(\frac{g(z)}{z^p}\right)^{\mu} \prec \left(\frac{zf''(z)}{pf(z)}\right)\left(\frac{f(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{g(z)}{z^p}\right)^{\mu} \prec \left(\frac{f(z)}{z^p}\right)^{\mu},$$

*and the function $\left(\frac{g(z)}{z^p}\right)^{\mu}$ is the best dominant.*

Combining Theorems 1 and 2, we obtain the following "sandwich-type result".

**Theorem 3.** *Let $f, g_i \in A_p$ $(j=1,2)$ and let*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta, \tag{2.27}$$

*where*

$$\phi_j(z) = \left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} g_j(z)}{U_{0,z}^{\alpha,\beta,\gamma} g_j(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_j(z)}{z^p}\right)^{\mu} \tag{2.28}$$
$$\left(-\infty<\alpha<1; -\infty<\beta<1; \gamma\in\mathbb{R}^+; \mu>0; z\in U\right),$$

*and $\delta$ is given by (2.3). If the function $\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} f(z)}{U_{0,z}^{\alpha,\beta,\gamma} f(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu}$ is univalent in $U$ and $\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu} \in Q$, then the condition*

$$\left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} g_1(z)}{U_{0,z}^{\alpha,\beta,\gamma} g_1(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} f(z)}{U_{0,z}^{\alpha,\beta,\gamma} f(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu}$$
$$\prec \left(\frac{U_{0,z}^{\alpha+1,\beta+1,\gamma+1} g_2(z)}{U_{0,z}^{\alpha,\beta,\gamma} g_2(z)}\right)\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_2(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{U_{0,z}^{\alpha,\beta,\gamma} f(z)}{z^p}\right)^{\mu} \prec \left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_1(z)}{z^p}\right)^{\mu},$$

*and the function $\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_1(z)}{z^p}\right)^{\mu}$ and $\left(\frac{U_{0,z}^{\alpha,\beta,\gamma} g_2(z)}{z^p}\right)^{\mu}$ are, respectively, the best subordinant and the best dominant.*

By taking $\alpha=\beta$ in Theorem 3 and using the relation (1.12) we get the following Corollary

**Corollary 5.** *Let $f, g_i \in A_p$ $(j=1,2)$ and let*

$$\Re\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta, \tag{2.29}$$

*where*

$$\phi_j(z) = \left(\frac{\Omega_z^{(\alpha+1,p)} g_j(z)}{\Omega_z^{(\alpha,p)} g_j(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} g_j(z)}{z^p}\right)^{\mu} \left(-\infty<\alpha<1; \mu>0; z\in U\right), \tag{2.30}$$

*and $\delta$ is given by (2.3). If the function $\left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu}$ is univalent in $U$ and $\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \in Q$, then the condition*

$$\left(\frac{\Omega_z^{(\alpha+1,p)} g_1(z)}{\Omega_z^{(\alpha,p)} g_1(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha+1,p)} f(z)}{\Omega_z^{(\alpha,p)} f(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu}$$

$$\prec \left(\frac{\Omega_z^{(\alpha+1,p)} g_2(z)}{\Omega_z^{(\alpha,p)} g_2(z)}\right)\left(\frac{\Omega_z^{(\alpha,p)} g_2(z)}{z^p}\right)^{\mu}, \tag{2.31}$$

*implies that*

$$\left(\frac{\Omega_z^{(\alpha,p)} g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha,p)} f(z)}{z^p}\right)^{\mu} \prec \left(\frac{\Omega_z^{(\alpha,p)} g_1(z)}{z^p}\right)^{\mu},$$

*and the function* $\left(\frac{\Omega_z^{(\alpha,p)} g_1(z)}{z^p}\right)^{\mu}$ *and* $\left(\frac{\Omega_z^{(\alpha,p)} g_2(z)}{z^p}\right)^{\mu}$ *are, respectively, the best subordinant and the best dominant.*

By taking $\alpha = 0$ in Corollary 5 and using the relation (1.13) and (1.14) we get the following Corollary.

**Corollary 6.** *Let* $f, g_i \in A_p$ $(j = 1, 2)$ *and let*

$$\Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta, \tag{2.32}$$

*where*

$$\phi_j(z) = \left(\frac{zg_j'(z)}{pg_j(z)}\right)\left(\frac{g_j(z)}{z^p}\right)^{\mu} (\mu > 0;\, z \in U), \tag{2.33}$$

*and* $\delta$ *is given by* (2.3). *If the function* $\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{f(z)}{z^p}\right)^{\mu}$ *is univalent in* $U$ *and* $\left(\frac{f(z)}{z^p}\right)^{\mu} \in Q$, *then the condition*

$$\left(\frac{zg_1'(z)}{pg_1(z)}\right)\left(\frac{g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{f(z)}{z^p}\right)^{\mu} \prec \left(\frac{zg_2'(z)}{pg_2(z)}\right)\left(\frac{g_2(z)}{z^p}\right)^{\mu},$$

*implies that*

$$\left(\frac{g_1(z)}{z^p}\right)^{\mu} \prec \left(\frac{f(z)}{z^p}\right)^{\mu} \prec \left(\frac{g_1(z)}{z^p}\right)^{\mu},$$

*and the function* $\left(\frac{g_1(z)}{z^p}\right)^{\mu}$ *and* $\left(\frac{g_2(z)}{z^p}\right)^{\mu}$ *are, respectively, the best subordinant and the best dominant.*


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Department of Mathematics, Alazhar University-Gaza, P. O. Box 1277, Gaza, Palestine.

E-mail: shenanjm @yahoo.com