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A NOTE ON HADAMARD TYPE INTEGRAL INEQUALITIES INVOLVING SEVERAL LOG-CONVEX FUNCTIONS

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Abstract. In this note, two new inegral inequalities of Hadamard type involving several differentiable log-convex functions are given. Two refinements of Hadamard's integral inequality for log-convex functions recently established by Dragomir are shown to be recaptured as special instances.

1. Introduction.

The following inequality is well known in the literature as Hadamard's inequality (see [5, p.137]):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to R$ is a convex function on the interval I of real numbers and $a,\ b \in I$ with $a < b$. In [1] Dragomir and Mond have proved that the following inequalities of Hadamard type hold:

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{b-a}\int_a^b \ln[f(x)]dx\right) \le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx$$
$$\le \frac{1}{b-a}\int_a^b f(x)dx \le L(f(a), f(b)) \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : I \to (0,\infty)$ is a log-convex function, $G(p,q) := \sqrt{pq}$ is the Geometric mean and $L(p,q) := \frac{p-q}{\ln p - \ln q}$ $(p \ne q)$ is the Logarithmic mean of the positive real numbers p, q (for $p = q$, we put $L(p,p) = p$). Recently in [3] Dragomir has established the following interesting refinements of Hadamard's inequalities for log-convex functions.

Let $f : I \to (0,\infty)$ be a differentiable log-convex function on the interval of real numbers $\overset{0}{I}$ (the interior of I) and $a,\ b \in \overset{0}{I}$ with $a < b$. Then the following inequalities

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hold:

$$\frac{\frac{1}{b-a}\int_a^b f(x)dx}{f\left(\frac{a+b}{2}\right)} \geq L\left(\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\right) \geq 1, \tag{1.3}$$

and

$$\frac{\frac{f(a)+f(b)}{2}}{\frac{1}{b-a}\int_a^b f(x)dx} \geq 1 + \log\left[\frac{\int_a^b f(x)dx}{\int_a^b f(x)\exp\left[\frac{f'(x)}{f(x)}\left(\frac{a+b}{2}-x\right)\right]dx}\right] \geq 1 + \log\left[\frac{\frac{1}{b-a}\int_a^b f(x)dx}{f\left(\frac{a+b}{2}\right)}\right] \geq 1. \tag{1.4}$$

For some recent results related to the Hadamard's inequality, see the books [2, 4, 5] where further references are given. The main purpose of this note is to establish the general versions of the inequalities (1.3) and (1.4) involving several differentiable log-convex functions. The method employed in our analysis is based on the basic properties of logarithms and the application of the well known Jensen's integral inequality.

2. Main Results

we start with the following theorem.

Theorem 1. *Let $f_i : I \to (0,\infty)$ $(i = 1,\dots,n)$ be differentiable log-convex functions on the interval of real numbers $\overset{0}{I}$ (the interior of I) and a, $b \in \overset{0}{I}$ with $a < b$. Then the following inequalities hold:*

$$\frac{\frac{1}{b-a}\int_a^b \prod_{i=1}^n f_i(x)dx}{\prod_{i=1}^n f_i\left(\frac{a+b}{2}\right)} \geq L\Bigg(\exp\left[\sum_{i=1}^n \frac{f_i'\left(\frac{a+b}{2}\right)}{f_i\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right], \exp\left[-\sum_{i=1}^n \frac{f_i'\left(\frac{a+b}{2}\right)}{f_i\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]\Bigg) \geq 1. \tag{2.1}$$

Proof. Since f_i $(i = 1,\dots,n)$ are differentiable and log-convex on $\overset{0}{I}$, we have that

$$\log f_i(x) - \log f_i(y) \geq \frac{d}{dy}(\log f_i(y))(x-y),$$

i.e.,

$$\log f_i(x) - \log f_i(y) \geq \frac{f_i'(y)}{f_i(y)}(x-y), \tag{2.2}$$

for all $x, y \in \overset{0}{I}$. Writing (2.2) for $i = 1, \ldots, n$, adding the resulting inequalities and using the properties of log it is easy to observe that

$$\log \left[\frac{\prod_{i=1}^{n} f_i(x)}{\prod_{i=1}^{n} f_i(y)} \right] \geq \sum_{i=1}^{n} \frac{f_i'(y)}{f_i(y)}(x-y), \tag{2.3}$$

for all $x, y \in \overset{0}{I}$. From (2.3) we have

$$\prod_{i=1}^{n} f_i(x) \geq \left(\prod_{i=1}^{n} f_i(y) \right) \exp \left[\sum_{i=1}^{n} \frac{f_i'(y)}{f_i(y)}(x-y) \right], \tag{2.4}$$

for $x, y \in \overset{0}{I}$. By taking $y = \frac{a+b}{2}$ in (2.4) we get

$$\frac{\prod_{i=1}^{n} f_i(x)}{\prod_{i=1}^{n} f_i\left(\frac{a+b}{2}\right)} \geq \exp \left[\sum_{i=1}^{n} \frac{f_i'\left(\frac{a+b}{2}\right)}{f_i\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2} \right) \right], \quad x \in [a,b]. \tag{2.5}$$

Integrating (2.5) over x on $[a,b]$ and using Jensen's integral inequality for exp (.) functions, we have

$$\begin{aligned} \frac{\frac{1}{b-a}\int_a^b \prod_{i=1}^{n} f_i(x)dx}{\prod_{i=1}^{n} f_i\left(\frac{a+b}{2}\right)} &\geq \frac{1}{b-a}\int_a^b \exp \left[\sum_{i=1}^{n} \frac{f_i'\left(\frac{a+b}{2}\right)}{f_i\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2} \right) \right] dx \\ &\geq \exp \left[\frac{1}{b-a}\int_a^b \sum_{i=1}^{n} \frac{f_i'\left(\frac{a+b}{2}\right)}{f_i\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2} \right) dx \right] = 1. \end{aligned} \tag{2.6}$$

Now by evaluating the middle integral in (2.6) as in [3, p.529] with suitable modifications, we get the required inequality in (2.1). The proof is complete.

The following inequality also holds.

Theorem 2. *Let f_i be as in Theorem 1. Then the following inequalities hold:*

$$\frac{\int_a^b \left(\prod_{i=1}^{n} f_i(y) \right) \left[\sum_{i=1}^{n} \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y \right) \right] dy}{\int_a^b \left(\prod_{i=1}^{n} f_i(y) \right) dy}$$

$$\le \log \left[\frac{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) \exp\left[\left[\sum_{i=1}^n \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y\right)\right] dy\right]}{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) dy} \right]$$

$$\le \left[\frac{\left(\prod_{i=1}^n f_i\left(\frac{a+b}{2}\right)\right)}{\frac{1}{b-a}\int_a^b \left(\prod_{i=1}^n f_i(y)\right) dy} \right]. \tag{2.7}$$

Proof. by taking $x = \frac{a+b}{2}$ in the inequality (2.4) we have

$$\prod_{i=1}^n f_i\left(\frac{a+b}{2}\right) \ge \left(\prod_{i=1}^n f_i(y)\right) \exp\left[\sum_{i=1}^n \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y\right)\right], \tag{2.8}$$

for all $y \in [a, b]$. Integrating (2.8) over y and using Jensen's integral inequality for exp(.) functions, we have

$$(b-a)\left(\prod_{i=1}^n f_i\left(\frac{a+b}{2}\right)\right)$$

$$\ge \int_a^b \left(\prod_{i=1}^n f_i(y)\right) \exp\left[\sum_{i=1}^n \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y\right)\right] dy$$

$$\ge \int_a^b \left(\prod_{i=1}^n f_i(y)\right) dy \exp\left[\frac{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) \left[\sum_{i=1}^n \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y\right)\right] dy}{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) dy} \right]. \tag{2.9}$$

From (2.9) we have

$$\exp\left[\frac{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) \left[\sum_{i=1}^n \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y\right)\right] dy}{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) dy} \right]$$

$$\le \frac{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) \exp\left[\sum_{i=1}^n \frac{f_i'(y)}{f_i(y)} \left(\frac{a+b}{2} - y\right)\right] dy}{\int_a^b \left(\prod_{i=1}^n f_i(y)\right) dy}$$

$$\le \frac{(b-a)\left(\prod_{i=1}^{n} f_i\left(\frac{a+b}{2}\right)\right)}{\int_a^b \left(\prod_{i=1}^{n} f_i(y)\right) dy},$$

which is equivalent to

$$\frac{\int_a^b \left(\prod_{i=1}^{n} f_i(y)\right)\left[\sum_{i=1}^{n} \frac{f_i'(y)}{f_i(y)}\left(\frac{a+b}{2}-y\right)\right] dy}{\int_a^b \left(\prod_{i=1}^{n} f_i(y)\right) dy}$$

$$\le \log\left[\frac{\int_a^b \left(\prod_{i=1}^{n} f_i(y)\right)\exp\left[\sum_{i=1}^{n} \frac{f_i'(y)}{f_i(y)}\left(\frac{a+b}{2}-y\right)\right] dy}{\int_a^b \left(\prod_{i=1}^{n} f_i(y)\right) dy}\right]$$

$$\le \log\left[\frac{\left(\prod_{i=1}^{n} f_i\left(\frac{a+b}{2}\right)\right)}{\frac{1}{b-a}\int_a^b \left(\prod_{i=1}^{n} f_i(y)\right) dy}\right]. \tag{2.10}$$

This is the desired inequality in (2.7) and the proof is complete.

It is interesting to note that by taking $i = 1$ and $f_1 = f$ in Theorem 1 we get (1.3). Further, by taking $i = 1$ and $f_1 = f$ in Theorem 2 and making the elementary calculations as in [3, p.530] we recapture the inequality (1.4).

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