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INTERVAL CRITERIA FOR OSCILLATION OF SECOND ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. Interval oscillation criteria are established for second order half-linear differential equations with damping that are different from most known ones in the sense that they are based only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. The results extend the integral averaging technique and include earlier results.

1. Introduction

This paper is concerned with the second order half-linear differential equation with damping

$$(r(t)\varphi_\alpha(x'(t)))' + p(t)\varphi_\alpha(x'(t)) + q(t)f(x(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $r \in C^1(I, \mathbb{R}^+)$, p, $q \in C(I, \mathbb{R})$, $\varphi_\alpha(u)$ is a real-valued function defined by $\varphi_\alpha(u) = |u|^{\alpha-2}u$, $\alpha > 1$ a fixed real number. $f \in C^1(\mathbb{R}, \mathbb{R})$ with $xf(x) > 0$ whenever $x \neq 0$, and satisfies $f'(x)/[f(x)]^{\frac{\alpha-2}{\alpha-1}} \geq \varepsilon > 0$ for $x \neq 0$, where $I = [t_0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$.

By a solution of Eq.(1.1) we mean a function $x(t) \in C^1([T_x, \infty), \mathbb{R})$ for some $T_x \geq t_0$, which has the property that $r(t)\varphi_\alpha(x'(t)) \in C^1([T_x, \infty), \mathbb{R})$ and satisfies Eq.(1.1) on $[T_x, \infty)$. We restrict our attention only to the nontrivial solution $x(t)$ of Eq.(1.1), i.e., to the solution $x(t)$ such that $\sup_{t>t_0}\{|x(t)|\} > 0$. A nontrivial solution of Eq.(1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Equation (1.1) is oscillatory if all its solutions are oscillatory.

Equation (1.1) can be considered as a natural generalization of the linear equation

$$(\, r(t)x'(t)\,)' + q(t)x(t) = 0 \tag{1.2}$$

and nonlinear equation with damping

$$x''(t) + p(t)x'(t) + q(t)f(x(t)) = 0 \tag{1.3}$$

and more general nonlinear equation with damping

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0 \tag{1.4}$$

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and half-linear differential equation

$$(r(t)|x'(t)|^{\alpha-2}x'(t))' + q(t)|x(t)|^{\alpha-2}x(t) = 0, \tag{1.5}$$

which have been the subject of intensive studies in recent years. Taking advantage of the averaging techniques similar to that exploited by Kamenev [3], Phiols [9] and Yan [11], a great deal of oscillation criteria for Eqs (1.2)-(1.5) has been obtained (see, for example, [1-4, 8-15]) and the references quoted therein). However, all above mentioned oscillation results involve the integral of the coefficients, and hence require the information of p and q on the entire half-linear $[t_0, \infty)$. Nevertheless, form the Sturmian Separation Theorem, oscillation is only an interval property, i.e., for each given solution of Eq.(1.1), if we can find a sequence of interval $[a_i, b_i]$, $a_i \to \infty$, $b_i < a_{i+1}$, such that the given solution has at least one zero in (a_i, b_i), then the solution is oscillatory.

Using the thoughts mentioned above, in 1999, Kong [5] employed the technique in the work of Philos [9] for the second order linear ordinary differential equation, and presented several interval oscillation criteria for Eq.(1.2) involving the Kamenev's type condition. Recently, Kong's results were further extended to the damped equation for Eq.(1.3) and Eq.(1.4) by Zhang [16] and Li and Agarwal [6], respectively. Therefore, it is natural to ask if it is possible to extend Kong's theorem to Eq.(1.5). An affirmative answer to this problem was given for the first time by Li [7] who obtained interval criteria for oscillation of Eq.(1.5). However, the results cannot be applied to damped half-linear differential equation (1.1).

Motivated by the ideas of Kong [5] and Wong [14], in this paper, we obtain several interval criteria for oscillation of Eq.(1.1), that is, criteria given by the behavior of Eq.(1.1) (or r, p, q and f) only on a sequence of subintervals of $[t_0, \infty)$. Our results improve and extend the results of Kong [5], Li [7], and Zhang [16], and can be applied to extreme cases such as $\int_{t_0}^{\infty} q(s)\,ds = -\infty$. Our methodology is somewhat different from that of previous authors. We believe that our approach is simpler and also provides a more unified account of study of Kamenev-type oscillation theorems [3]. In particular, several examples that dwell upon the sharp conditions of our results are also included. We will also show that we do not impose conditions on the sign of the functions p and q.

2. Main Results

In the sequel, we say that a function $H(t, s)$ belongs to a function class $\mathcal{H}$, denoted by $H \in \mathcal{H}$, if $H \in C(D, [0, \infty)\,)$, where $D = \{(t, s) : -\infty < s \le t < \infty\}$, which satisfies

$$H(t, t) = 0, \quad H(t, s) > 0 \quad \text{for} \quad t > s \ge t_0.$$

Further, H has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that, for h_1, $h_2 \in C_{loc}(D, \mathbb{R})$

$$\frac{\partial H}{\partial t}(t, s) = h_1(t, s)H(t, s) \quad \text{and} \quad \frac{\partial H}{\partial s}(t, s) = -h_2(t, s)H(t, s).$$

Note that $(t-s)^n$ for $n > 1$; $\ln t/s$; and $a(t-s)$ with $a \in C^1(0,\infty)$, $a'(t) > 0$ for $t > 0$ and $a(0) = 0$ belong to function class $\mathcal{H}$. Particularly, if $H(t,s) = H(t-s) \in \mathcal{H}$, then $h_1(t,s) = h_2(t,s) =: h(t-s)$. We denote $H \in \mathcal{H}_0$ for all the functions $H(t-s) \in \mathcal{H}$.

Let $k \in C^1(I, \mathbb{R}^+)$, we take operators $A_T^k(\cdot, t)$ and $B_T^k(\cdot, t)$ which are defined in [14], in term of H and k as, for $\phi \in C(I, \mathbb{R})$

$$A_T^k(\phi, t) = \int_T^t H(s,T)\phi(s)k(s)\,ds, \quad t \geq T \tag{2.1}$$

and

$$B_T^k(\phi, t) = \int_T^t H(t,s)\phi(s)k(s)\,ds, \quad t \geq T. \tag{2.2}$$

It is easy to verify that $A_T^k(\cdot, t)$ and $B_T^k(\cdot, t)$ are linear operators and satisfy

$$A_T^k(\phi', t) = H(t,T)\phi(t)k(t) - A_T^k([h_1 + \frac{k'}{k}]\phi, t), \quad t \geq T \tag{2.3}$$

and

$$B_T^k(\phi', t) = -H(t,T)\phi(T)k(T) + B_T^k([h_2 - \frac{k'}{k}]\phi, t), \quad t \geq T, \tag{2.4}$$

where $\phi \in C^1(I, \mathbb{R})$, $h_1 = h_1(s,t)$ and $h_2 = h_2(t,s)$.

To simplify notations, we define, for an arbitrary given function $\rho \in C^1(I, \mathbb{R}^+)$, for $(t,s) \in D$

$$\lambda_1 = \lambda_1(s,t) = h_1(s,t) + \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)},$$
$$\lambda_2 = \lambda_2(t,s) = h_2(t,s) - \frac{\rho'(s)}{\rho(s)} + \frac{p(s)}{r(s)}$$

and

$$k = \varepsilon^{1-\alpha}\alpha^{-\alpha}(\alpha-1)^{\alpha-1}.$$

Theorem 2.1. *Suppose that there exist functions $H \in \mathcal{H}$ and $\rho \in C^1(I, \mathbb{R}^+)$. If for each $T \geq t_0$, there exist a, b and c such that $T \leq a < c < b$ and*

$$\frac{1}{H(c,a)} A_a^\rho(\, q - k\,r|\lambda_1|^\alpha, c\,) + \frac{1}{H(b,c)} B_c^\rho(\, q - k\,r|\lambda_2|^\alpha, b\,) > 0, \tag{2.5}$$

then Eq.(1.1) is oscillatory.

Proof. Suppose to the contrary that there exists a solution $x(t)$ of Eq.(1.1) which is not oscillatory. In view of the assumption that $xf(x) > 0$ whenever $x \neq 0$, we may suppose that $x(t) > 0$ for $t \geq t_0$. Define

$$w(t) = \frac{r(t)\varphi_\alpha(x'(t))}{f(x(t))}, \quad t \geq t_0.$$

By Eq.(1.1), we can find that $w(t)$ satisfies the following differential inequality

$$w'(t) \leq -q(t) - \frac{p(t)}{r(t)}w(t) - \varepsilon[r(t)]^{1-\beta}|w(t)|^{\beta}, \tag{2.6}$$

where $1/\alpha + 1/\beta = 1$. For each $T \geq t_0$, there exist a, b and c such that $T \leq a < c < b$. Applying the operator $A_t^{\rho}(\,\cdot, c)$ to (2.6) and using (2.3), we have

$$A_t^{\rho}(q, c) \leq -H(c,t)\rho(c)w(c) + A_t^{\rho}(\,\lambda_1, c) - \varepsilon A_t^{\rho}(r^{1-\beta}|w|^{\beta}, c\,). \tag{2.7}$$

Using Young's inequality

$$|\lambda_1||w| \leq kr|\lambda_1|^{\alpha} + \varepsilon r^{1-\beta}|w|^{\beta}. \tag{2.8}$$

Substituting (2.8) in (2.7), it yields

$$\frac{1}{H(c,t)}A_t^{\rho}(\,q - k\,r|\lambda_1|^{\alpha}, c\,) \leq -\rho(c)w(c),$$

let $t \to a^+$, we have

$$\frac{1}{H(c,a)}A_a^{\rho}(\,q - k\,|\lambda_1|^{\alpha}, c\,) \leq -\rho(c)w(c). \tag{2.9}$$

On the other hand, similar to that of the proof of (2.9), we obtain

$$\frac{1}{H(b,c)}B_c^{\rho}(\,q - k\,r|\lambda_2|^{\alpha}, b\,) \leq \rho(c)w(c). \tag{2.10}$$

Then, (2.9) and (2.10) imply the desired contradiction.

Remark 2.1. Theorem 2.1 improves Theorem 2.1 in [5], Theorem 2.2 in [7] and Theorem 1 in [16].

Theorem 2.2. *Suppose that there exist functions $H \in \mathcal{H}$ and $\rho \in C^1(I, \mathbb{R}^+)$. If for each $T \geq t_0$, $t > T$*

$$\limsup_{t\to\infty} A_T^{\rho}(q - kr|\lambda_1|^{\alpha}, t) > 0 \tag{2.11}$$

and

$$\limsup_{t\to\infty} B_T^{\rho}(q - kr|\lambda_2|^{\alpha}, t) > 0, \tag{2.12}$$

then Eq.(1.1) is oscillatory.

Proof. For any $T \geq t_0$, let $a = T$, in (2.11), we choose $T = a$, then there exists $c > a$ such that

$$A_a^{\rho}(q - kr|\lambda_1|^{\alpha}, c) > 0. \tag{2.13}$$

In (2.12), let $T = c$, then there exists $b > c$ such that

$$B_c^\rho(q - kr|\lambda_2|^\alpha, b) > 0. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain (2.5), Thus, the conclusion follows from Theorem 2.1.

Remark 2.2. Theorem 2.2 improves Corollary 2.4 in [5], Theorem 2.3 in [7] and Corollary 1 in [16].

Theorem 2.3. *Suppose that there exist functions $H \in \mathcal{H}_0$ and $\rho \in C^1(I, \mathbb{R}^+)$. If for each $t \geq t_0$ there exist $a, b \in \mathbb{R}$ such that $T \leq a < c$ and*

$$\begin{aligned}\int_a^c & H(s-a)\,[\,q(s)\rho(s) + q(2c-s)\rho(2c-s)\,]\,ds \\ &> k\int_a^c H(s-a)\left[r(s)|h(s-a) + \frac{\rho'(s)}{\rho(s)} - \frac{p(s)}{r(s)}|^\alpha \rho(s)\right. \\ &\left. +r(2c-s)|h(s-a) - \frac{\rho'(2c-s)}{\rho(2c-s)} + \frac{p(2c-s)}{r(2c-s)}|^\alpha \rho(2c-s)\right] ds,\end{aligned} \tag{2.15}$$

then Eq.(1.1) is oscillatory.

Proof. Let $b = 2c - a$. Then $H(b - c) = H(c - a)$, and for any function $g \in L[a, b]$, we have

$$\int_c^b g(s)\,ds = \int_a^c g(2c-s)\,ds.$$

Thus, (2.15) holds implies that (2.5) holds, therefore Eq.(1.1) is oscillatory by Theorem 2.1.

Remark 2.3. Theorem 2.3 includes Theorem 2.2 in [5], Theorem 2.4 in [7] and Corollary 2 in [16].

Let

$$H(t,s) = (t-s)^\lambda \quad \text{and} \quad \rho(t) = \exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}\,ds\right), \quad \text{for} \quad t \geq s \geq t_0,$$

where $\lambda \geq \alpha$ is a number. By Theorem 2.3, we have the following corollary.

Corollary 2.1. *Suppose that for each $T \geq t_0$, there exist a, c such that $T \leq a < c$ and*

$$\begin{aligned}\int_a^c & (s-a)^\lambda \left[q(s)\,\exp\left(\int_{t_0}^s \frac{p(\tau)}{r(\tau)}d\tau\right) + q(2c-s)\exp\left(\int_{t_0}^{2c-s} \frac{p(\tau)}{r(\tau)}d\tau\right)\right] ds \\ &> \alpha^\lambda k\int_a^c (s-a)^{\lambda-\alpha}\left[r(s)\exp\left(\int_{t_0}^s \frac{p(\tau)}{r(\tau)}\,d\tau\right)\right. \\ &\qquad\qquad \left. +r(2c-s)\exp\left(\int_{t_0}^{2c-s} \frac{p(\tau)}{r(\tau)}d\tau\right)\right] ds,\end{aligned} \tag{2.16}$$

where $\lambda \ge \alpha$*. Then Eq.(1.1) is oscillatory.*

Define

$$\rho(t) = \exp\left(\int_{t_0}^{t} \frac{p(\tau)}{r(\tau)} d\tau\right), \quad R(t) = \int_{t_0}^{t} [\, r(s)\rho(s)\,]^{\frac{1}{1-\alpha}}\, ds$$

and

$$H(t,s) = [R(t) - R(s)]^{\lambda}, \quad t \ge s \ge t_0.$$

where $\lambda > \alpha - 1$ is a number.

Corollary 2.2. *Let* $\lim_{t\to\infty} R(t) = \infty$*. Assume that for each* $T \ge t_0$ *and for some* $\lambda > \alpha - 1$ *we have*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha+1}(t)} \int_T^t [R(s) - R(T)]^{\lambda} q(s) \exp\left(\int_{t_0}^{s} \frac{p(\tau)}{r(\tau)}\, d\tau\right) ds > \frac{k\lambda^{\alpha}}{\lambda - \alpha + 1} \tag{2.17}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha+1}(t)} \int_T^t [R(t) - R(s)]^{\lambda} q(s) \exp\left(\int_{t_0}^{s} \frac{p(\tau)}{r(\tau)}\, d\tau\right) ds > \frac{k\lambda^{\alpha}}{\lambda - \alpha + 1}, \tag{2.18}$$

then Eq.(1.1) *is oscillatory.*

Proof. Noting that

$$h_1(t,s) = \lambda [R(t) - R(s)]^{-1} [\, r(t)\rho(t)\,]^{\frac{1}{1-\alpha}}$$

and

$$h_2(t,s) = \lambda [R(t) - R(s)]^{-1} [\, r(s)\rho(s)\,]^{\frac{1}{1-\alpha}},$$

then

$$A_T^{\rho}(\, r|\lambda_1|^{\alpha}, t) = \frac{\lambda^{\alpha}}{\lambda - \alpha + 1} [R(t) - R(T)]^{\lambda-\alpha+1} \tag{2.19}$$

and

$$B_T^{\rho}(\, r|\lambda_2|^{\alpha}, t) = \frac{\lambda^{\alpha}}{\lambda - \alpha + 1} [R(t) - R(T)]^{\lambda-\alpha+1}. \tag{2.20}$$

From (2.19) and $\lim_{t\to\infty} R(t) = \infty$, we have

$$\limsup_{t\to\infty} \frac{1}{[R(t)]^{\lambda-\alpha+1}} A_T^{\rho}(\, kr|\lambda_1|^{\alpha}, t) = \frac{k\lambda^{\alpha}}{\lambda - \alpha + 1}. \tag{2.21}$$

Thus, by (2.17) and (2.11), we obtain that (2.21) holds. Similarly, (2.18) and (2.20) imply that (2.12) holds. By Theorem 2.2, Eq.(1.1) is oscillatory.

Remark 2.4. Corollary 2.1 includes Corollary 3 in [16]; Corollary 2.2 improves Theorem 2.5 in [9], Theorem 2.3 in [5], Corollary 2.2 in [6].

Now, we give some examples to illustrate the efficiency of theorems.

Example 2.1. Consider the following equation

$$(t|x'(t)|^2x'(t))' - |x'(t)|^2x'(t) + q(t)x^3(t) = 0, \quad t \geq 1, \tag{2.22}$$

where $q(t)$ is given as follows

$$q(t) = \begin{cases} 7(t-3n)t, & \text{if } 3n \leq t \leq 3n+1, \\ 7(-t+3n+2)t, & \text{if } 3n+1 < t \leq 3n+2, \\ g(t), & \text{if } 3n+2 < t \leq 3n+3, \end{cases}$$

where $g(t)$ is an arbitrary function such that $q(t)$ is continuous, $n \in \mathbb{N}_0 = \{0.1, \cdots\}$. For any $T \geq 0$ there exists $n \in \mathbb{N}_0$ such that $3n \geq T$. Let $a = 3a$, $c = 3n+1$ and $H(t,s) = (t-s)^5$. It is easy to see that (2.16) holds and then Eq.(2.22) is oscillatory by Corollary 2.1. Nevertheless, $g(t)$ can be selected as a "bad" term of $q(t)$ such that $\int_1^\infty q(s)\,ds = -\infty$.

Example 2.2. Consider the nonlinear equation

$$(|x'(t)|^{\alpha-2}x'(t))' + \frac{1-\alpha}{t}|x'(t)|^{\alpha-2}x'(t) + \frac{\gamma}{t^\alpha}|x(t)|^{\alpha-2}x(t) = 0, \quad t \geq 1, \tag{2.23}$$

Then $R(t) = \frac{1}{2}(t^2-1)$. For $\lambda > \alpha - 1$,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha+1}(t)} \int_T^t [\,R(s) - R(T)\,]^\lambda q(s) \exp\left(\int_1^s \frac{p(\tau)}{r(\tau)}\,d\tau\right) ds \\ &\qquad = \limsup_{t\to\infty} \frac{\gamma}{R^{\lambda-\alpha+1}(t)} \int_T^t [\,R(s) - R(T)\,]^\lambda s^{1-2\alpha}\,ds = \frac{2^{-\alpha}\gamma}{\lambda-\alpha+1}. \end{aligned} \tag{2.24}$$

By Lemma 3.1 in [5], we have

$$\int_T^t [\,R(t) - R(s)\,]^\lambda s^{1-2\alpha}\,ds \geq \int_T^t [\,R(s) - R(T)\,]^\lambda s^{1-2\alpha}\,ds. \tag{2.25}$$

By (2.24) and(2.25), for $\gamma > (2(\alpha+1))^\alpha k$ (where $k = \alpha^{-\alpha}$), there is $\lambda > \alpha - 1$ such that $2^{-\alpha}\gamma/(\lambda-\alpha+1) > k\lambda^\alpha/(\lambda-\alpha+1)$. This means that (2.17) and (2.18) hold for the same λ. Applying Corollary 2.2, Eq. (2.23) is oscillatory if $\gamma > (2(\alpha+1))^\alpha k$.

Remark 2.5. The results in [7] cannot apply to Eq.(2.22) and (2.23).

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